

Artificial gauge fields for quantum gases

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Introduction

Magnetic phenomena play an essential role in quantum physics. Notions or phenomena as varied as gauge invariance, the quantum Hall effect, spin-orbit coupling, the Aharonov-Bohm effect, and topological insulators all originate in the interaction between moving charges and a magnetic field $\mathbf{B}$.

For a particle of mass M , charge q , and velocity $\mathbf{v}$, this interaction is described in terms of the Lorentz force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, or, equivalently, by the Hamiltonian

$$\hat{H} = \frac{(\hat{\mathbf{p}} - q\mathbf{A}(\hat{\mathbf{r}}))^2}{2M},$$

where $\mathbf{A}$ is the vector potential associated with the magnetic field $\mathbf{B}$. The study of magnetism with cold atoms constitutes an important part of the general program of quantum simulation based on these novel gases. But the electrical neutrality of atoms ($q = 0$) requires us to resort to “tricks”, for example, light beams of suitably chosen frequencies and directions, in order to reach situations equivalent to those encountered for the electron fluids of ordinary matter.

This year’s course and seminars will review this very active research, on both the theoretical and experimental fronts. The course, made up of seven lectures, will be organized as follows:

- The first lecture will be devoted to the basics of describing the motion of a particle in a magnetic field. Starting from the Lorentz force, we will derive the Hamiltonian approach to this problem, as well as the notion of gauge invariance, both at the level of electrodynamics and at the level of quantum mechanics. We will illustrate these concepts by discussing the Aharonov–Bohm effect, as well as the implication of hypothetical magnetic monopoles.

- The second lecture will be devoted to the study of the quantum motion of a charged particle in a uniform magnetic field. In particular, we will discuss the form of the energy spectrum, quantized into Landau levels. From this we will derive the concept of edge states, which play a crucial role in interpreting the quantum Hall effect.
- Lecture 3 will introduce the notion of the Berry phase and geometric gauge potentials. We will start from the adiabatic approximation to show how the Berry phase arises, and we will then make the connection with the notion of parallel transport. We will conclude with a Born-Oppenheimer-type approach for a particle with several internal states, for which the geometric gauge potentials, both vector and scalar, appear explicitly in a Schrödinger equation reduced to a single one of these internal states.
- Lectures 4 and 5 will explore two possible approaches to simulating magnetism with cold-atom gases: setting the gas into rotation, or using geometric phases induced by light. We will describe several recent experiments in this field, and we will discuss the limitations arising from heating due to the spontaneous emission of photons when the gauge field is induced by a light beam. We will also address the implementation of spin-orbit coupling, in both one and two dimensions.
- Lecture 6 will be devoted to the simulation of lattice magnetism. We will first discuss its main features, such as the fractal structure of the spectrum known as the Hofstadter butterfly. We will then review various techniques, such as “shaken” lattices or laser-assisted tunneling between different internal states, which make it possible to simulate this magnetism for atoms trapped in an optical lattice.
- Finally, lecture 7 will address the combined effect of orbital magnetism

and interactions between atoms. We will mainly be interested in systems described by a macroscopic wave function, in which vortices can be nucleated. We will show how these vortices arrange themselves into regular lattices (*Abrikosov lattices*), and we will conclude with a few remarks on the physics likely to appear at very high magnetic fields, with the emergence of strongly correlated states reminiscent of those of the quantum Hall effect.

Chapter I

The magnetism of a point particle

In this first chapter, we will begin by reviewing the key elements of magnetism, first from the point of view of the magnetic fields themselves via Maxwell's equations, then from the point of view of the motion of charged particles. We will successively address the classical formalism (Newtonian dynamics), then the quantum formalism. In the latter case, we will emphasize the notion of gauge transformation, both from the point of view of the electromagnetic field and from the point of view of the quantum formulation of the particles' motion.

In the second part of this chapter, we will present two illustrations of this formalism: we will analyze the Aharonov–Bohm effect, which sheds new light on the notion of the vector potential and which allows us to introduce the concept of geometric phase, which will play an essential role later in this course. We will also briefly discuss the notion of the magnetic monopole, revisiting a famous argument of Dirac on the link between the possible existence of such monopoles and the quantization of electric charge (Dirac 1931; Dirac 1948). We will conclude with a very brief presentation of a recent experiment simulating such a monopole within a Bose–Einstein condensate.

We will not address in this first chapter the important problem of the quantum motion of a charged particle in a magnetic field, with the Landau-level structure that plays a crucial role in phenomena related to orbital magnetism, such as the quantum Hall effect. This development will be the subject of chapter 2.

1 Some elements of magnetostatics

We will begin this chapter with a review of magnetostatics: starting from the basic equations satisfied by the magnetic field $\mathbf{B}$ in the time-independent case, our goal is to introduce the notion of the vector potential $\mathbf{A}$ and to discuss a few examples of geometries relevant for what follows.

1-1 Equations of magnetostatics

Here we are interested in the case of a magnetic field $\mathbf{B}$ that is independent of time but depends on the position $\mathbf{r}$. This field is computed as a function of the externally imposed currents from the two equations of magnetostatics. These equations can be seen as a particular case of Maxwell's equations for a time-independent electromagnetic field.

The first equation of magnetostatics is written

$$\nabla \cdot \mathbf{B} = 0 \tag{I.1}$$

and expresses the absence of magnetic charge (monopole): the flux of $\mathbf{B}$ through a closed surface is always zero¹.

¹We will come back to the consequences of the possible existence of magnetic monopoles. On a formal level, let us also note that a more correct way of describing known particles consists in saying that they all have the same ratio of *electric charge to magnetic charge*. A

The second equation relates the field $\mathbf{B}$ to the currents creating this field

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \quad (\text{I.2})$$

where we have (possibly) included in $\mathbf{j}$ the “microscopic currents” resulting from the magnetism of material media.

Remark: These two equations should be compared with those determining the electric field in electrostatics,

$$\nabla \cdot \mathbf{E} = \frac{\rho_{\text{elec.}}(\mathbf{r})}{\epsilon_0}, \quad \nabla \times \mathbf{E} = 0, \quad (\text{I.3})$$

where $\rho_{\text{elec.}}(\mathbf{r})$ is the electric charge density at point $\mathbf{r}$.

1-2 The vector potential

The first equation of magnetostatics, $\nabla \cdot \mathbf{B} = 0$, places a strong constraint on the possible form of the vector fields $\mathbf{B}(\mathbf{r})$ that can be realized. To account for this constraint in a simple way, one can show (Poincaré’s lemma) that the field $\mathbf{B}$ can always be written in the form

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}). \quad (\text{I.4})$$

It is clear that relation (I.4) does not define the vector potential $\mathbf{A}(\mathbf{r})$ uniquely. More precisely, two vector potentials $\mathbf{A}$ and $\mathbf{A}'$ such that $\nabla \times (\mathbf{A}' - \mathbf{A}) = 0$ will lead to the same magnetic field. Let us examine the consequences of this equation in a simply connected volume, the whole of space for example; the equation $\nabla \times \mathbf{X}(\mathbf{r}) = 0$ can be solved as $\mathbf{X}(\mathbf{r}) = \nabla \chi(\mathbf{r})$, where $\chi(\mathbf{r})$ is a scalar function of $\mathbf{r}$. We deduce that the set of vector potentials associated with a given magnetic field $\mathbf{B}$ forms an equivalence class, whose various members are related by a relation of the type

$$\mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \nabla \chi(\mathbf{r}). \quad (\text{I.5})$$

The passage from $\mathbf{A}(\mathbf{r})$ to $\mathbf{A}'(\mathbf{r})$ is called a *gauge transformation*, and the function $\chi(\mathbf{r})$ is the *gauge function* associated with this transformation.

rotation in the abstract spaces $\{\mathbf{E}, \mathbf{H}\}$ and $\{\mathbf{D}, \mathbf{B}\}$ then makes it possible to recover the usual formulation, which we use here [Jackson (1998), Chap. 6, § 6.11]

1-3 Examples

Below we present a few examples that will be useful for the rest of this course. We will use the notation $\{\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_z\}$ to denote a right-handed orthonormal frame of space, with $\mathbf{r} = (x, y, z)$. We will also use cylindrical coordinates (ρ, φ, z) and spherical coordinates (r, θ, φ) , with in both cases the azimuthal unit vector $\mathbf{u}_\varphi = \mathbf{u}_x \cos \varphi + \mathbf{u}_y \sin \varphi$.

Example 1: uniform field $\mathbf{B}$. Consider a magnetic field $\mathbf{B}$ uniform in space, oriented by convention along the z axis. A possible choice of vector potential (symmetric gauge) is

$$\mathbf{A}(\mathbf{r}) = \begin{pmatrix} -By/2 \\ +Bx/2 \\ 0 \end{pmatrix}. \quad (\text{I.6})$$

This vector potential can also be written in vector form

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2} \mathbf{B} \times \mathbf{r}. \quad (\text{I.7})$$

This choice of gauge gives (almost) symmetric roles to the variables x and y , which is satisfying given the symmetry of the problem. On the other hand, it breaks the translational invariance of the problem by singling out a point in space, the origin O , where the vector potential vanishes.

One often uses the gauge transformation associated with the function

$$\chi(\mathbf{r}) = \frac{B}{2} xy, \quad \nabla \chi = \frac{B}{2} (y \mathbf{u}_x + x \mathbf{u}_y) \quad (\text{I.8})$$

to obtain the vector potential in the Landau gauge²

$$\mathbf{A}'(\mathbf{r}) = \begin{pmatrix} 0 \\ +Bx \\ 0 \end{pmatrix}. \quad (\text{I.9})$$

This form breaks the symmetry between the x and y axes, but it is often convenient for calculations in quantum physics; we will come back to it in the next chapter.

²One can also take the gauge function $-\chi(\mathbf{r})$, which leads to $\mathbf{A}''(\mathbf{r}) = -By \mathbf{u}_x$.

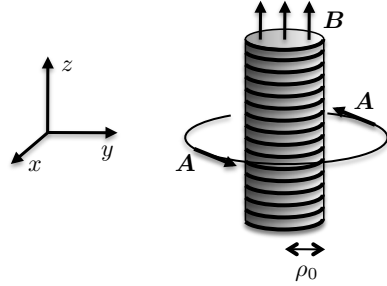


Figure I.1. Infinite solenoid with axis z , leading to a uniform field inside the solenoid and zero outside. The current in this solenoid is assumed to be azimuthal, flowing over a uniform sheet of radius r_0 . The gauge choice (I.10) leads to the azimuthal vector potential shown in the figure.

Example 2: infinite solenoid. Let us now consider the ideal, infinite solenoid of radius ρ_0 and axis z shown in figure I.1. The two equations of magnetostatics, together with translational invariance along the z axis and the assumption that the field vanishes as $\rho \rightarrow \infty$, lead to the following conclusions:

- Inside the solenoid, the field $\mathbf{B}$ is parallel to the z axis and uniform (with amplitude denoted below by B_0).
- The field $\mathbf{B}$ vanishes outside the solenoid.

One can check that this field $\mathbf{B}(\mathbf{r})$ can be obtained from an azimuthal vector potential $\mathbf{A}(\mathbf{r}) = A(\rho) \mathbf{u}_\varphi$ with, again in cylindrical coordinates:

$$A(\rho) = B_0 \frac{\rho_0^2}{2\rho} \quad \text{if } \rho > \rho_0, \quad A(\rho) = B_0 \frac{\rho}{2} \quad \text{if } \rho < \rho_0. \quad (\text{I.10})$$

One often uses the limit in which the radius ρ_0 tends to 0, with the field value B_0 tending to infinity so as to keep constant the flux $\Phi = \pi \rho_0^2 B_0$ of the magnetic field through a plane perpendicular to the z axis. This is then referred to as a *flux line*, corresponding to

$$\mathbf{A}(\mathbf{r}) = \frac{\Phi}{2\pi\rho} \mathbf{u}_\varphi, \quad \mathbf{B}(\mathbf{r}) = \Phi \delta(x)\delta(y) \mathbf{u}_z. \quad (\text{I.11})$$

One immediately checks that the circulation of $\mathbf{A}$ around any circle centered on the z axis and parallel to the xy plane equals Φ , just as does the flux of $\mathbf{B}$ through the corresponding disk. Note also that the vector potential becomes singular on the z axis in this limit of an infinitely thin solenoid. In what follows, it will be useful to write this same vector potential in spherical coordinates:

$$\mathbf{A}(\mathbf{r}) = \frac{\Phi}{2\pi} \frac{1}{r \sin \theta} \mathbf{u}_\varphi. \quad (\text{I.12})$$

Example 3: magnetic monopole! The vector potential was introduced to account for the constraint $\nabla \cdot \mathbf{B} = 0$, expressing the absence of magnetic charges. If one is willing to work with a vector potential exhibiting singularities similar to the one appearing in the case of the infinitely narrow solenoid [eq. (I.12)], one can nevertheless (almost) describe an isolated magnetic monopole with the help of this vector potential.

Consider the vector potential in spherical coordinates

$$\mathbf{A}_m^{(1)}(\mathbf{r}) = \frac{q_m}{4\pi} \frac{1 - \cos \theta}{r \sin \theta} \mathbf{u}_\varphi. \quad (\text{I.13})$$

This potential is defined at every point in space except on the negative part of the z axis, for which $\theta = \pi$. If we apply the standard differentiation formulas to compute $\nabla \times \mathbf{A}_{m,1}$, valid *a priori* away from the singularity of the half-axis $z < 0$, we find

$$B_r = (\nabla \times \mathbf{A})_r = \frac{1}{r \sin \theta} \left(\frac{\partial(A_\varphi \sin \theta)}{\partial \theta} - \frac{\partial A_\theta}{\partial \varphi} \right) = \frac{q_m}{4\pi r^2} \quad (\text{I.14})$$

$$B_\theta = (\nabla \times \mathbf{A})_\theta = \frac{1}{r \sin \theta} \left(\frac{\partial A_r}{\partial \varphi} - \sin \theta \frac{\partial(r A_\varphi)}{\partial r} \right) = 0 \quad (\text{I.15})$$

$$B_\varphi = (\nabla \times \mathbf{A})_\varphi = \frac{1}{r} \left(\frac{\partial(r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right) = 0 \quad (\text{I.16})$$

It thus seems that we have succeeded in generating the magnetic field of a monopole

$$\mathbf{B}_m = \frac{q_m}{4\pi r^2} \mathbf{u}_r \quad (\text{I.17})$$

from the equation $\mathbf{B} = \nabla \times \mathbf{A}$, which seems paradoxical! The resolution of this paradox appears when we compute the circulation of $\mathbf{A}$ around a

small circle with axis z , of radius $r \sin \theta$; one finds

$$\oint (\mathbf{A} \cdot \mathbf{u}_\varphi) d\varphi = \frac{q_m}{2} (1 - \cos \theta). \quad (\text{I.18})$$

As the radius of the circle tends to 0, one finds a vanishing circulation for a circle located on the half-axis $z > 0$ ($\theta \rightarrow 0$) and a circulation equal to q_m for a circle located on the half-axis $z < 0$ ($\theta \rightarrow \pi$). This means that, in addition to the monopole field (I.17), the vector potential (I.13) generates a field parallel to the z axis and localized on the half-axis $z < 0$, like that of the solenoid considered previously. The total magnetic field associated with the vector potential (I.13) is thus

$$\mathbf{B}^{(1)}(\mathbf{r}) = \frac{q_m}{4\pi r^2} \mathbf{u}_r + q_m \Theta(-z) \delta(x) \delta(y) \mathbf{u}_z, \quad (\text{I.19})$$

where $\Theta(z)$ is the Heaviside function³. One says that a *Dirac string* is attached to the monopole of charge q_m , starting from the point where the monopole is located ($r = 0$) and extending to $z = -\infty$.

One can also consider the vector potential

$$\mathbf{A}_m^{(2)}(\mathbf{r}) = -\frac{q_m}{4\pi} \frac{1 + \cos \theta}{r \sin \theta} \mathbf{u}_\varphi. \quad (\text{I.20})$$

which has a singularity on the half-axis $z > 0$. The corresponding magnetic field contains the same monopole as (I.19), but the singularity line is reversed:

$$\mathbf{B}^{(2)}(\mathbf{r}) = \frac{q_m}{4\pi r^2} \mathbf{u}_r - q_m \Theta(z) \delta(x) \delta(y) \mathbf{u}_z. \quad (\text{I.21})$$

Note that the vector potential (I.13) can in practice be obtained by considering a chain of magnetic dipoles aligned along the half-axis $z < 0$ (figure I.1b), or else a semi-infinite solenoid of arbitrarily small radius (figure I.1c) (Jackson 1998). For (I.20), the chain of dipoles extends along the half-axis $z > 0$, from $z = 0$ to $z = +\infty$.

Let us also note that the difference between the two vector potentials $\mathbf{A}_m^{(1)}$ and $\mathbf{A}_m^{(2)}$ is exactly equal to the vector potential proposed in (I.12) to describe an infinite solenoid, with $\Phi = q_m$, which is consistent with the fact that these vector potentials each correspond to a semi-infinite solenoid, one

along $z > 0$, the other along $z < 0$, with currents flowing in opposite directions. This shows that $\mathbf{A}_m^{(1)}$ and $\mathbf{A}_m^{(2)}$ are not related by a gauge transformation, at least over the whole space R^3 , since $\mathbf{A}_m^{(1)} - \mathbf{A}_m^{(2)} \neq \nabla \chi$. We will see a little further on how Dirac exploited this point to deduce the quantization of electric charge from the possible existence of a magnetic monopole.

2 Classical particle in a magnetic field

We now turn to the description of the motion of a charged point particle, described by classical mechanics, in a magnetic field $\mathbf{B}(\mathbf{r})$. In what follows, we will restrict ourselves to the non-relativistic case, the velocity of the material particles being assumed to be very small compared to the speed of light.

2-1 Lorentz force and cyclotron motion

Let M denote the mass of the particle and q its charge. Our starting point will be the Lorentz force acting on the particle of velocity $\dot{\mathbf{r}} = \mathbf{v}$:

$$\mathbf{F}_L = q \mathbf{v} \times \mathbf{B}, \quad (\text{I.22})$$

leading to the equation of motion

$$M \ddot{\mathbf{r}} = q \dot{\mathbf{r}} \times \mathbf{B}. \quad (\text{I.23})$$

Particular case. If the field $\mathbf{B}$ is uniform, i.e., independent of the point of space considered, the motion of the particle in the plane perpendicular to $\mathbf{B}$ is uniform circular motion (cyclotron motion), with angular frequency

$$\omega_c = qB/M. \quad (\text{I.24})$$

The motion is uniform and rectilinear along the axis of the field $\mathbf{B}$.

To prepare the ground for the quantum description, we will now express the fundamental law of dynamics (I.23) within the Lagrangian and Hamiltonian formalisms.

³ $\Theta(z) = 1$ if $z > 0$, $\Theta(z) = 0$ if $z < 0$, $\Theta(0) = 1/2$.

2-2 Euler–Lagrange equations

For a point particle, the Lagrangian formalism amounts to specifying a function $L(\mathbf{r}, \dot{\mathbf{r}}, t)$ to describe the dynamics of the particle. From this Lagrange function (or *Lagrangian*), one computes, for any continuous trajectory in the space $(\mathbf{r}(t), \dot{\mathbf{r}}(t))$, the action

$$S = \int_{t_1}^{t_2} L[\mathbf{r}(t), \dot{\mathbf{r}}(t), t] dt. \quad (\text{I.25})$$

We posit that the trajectory actually followed by the particle to go from the starting point $\mathbf{r}(t_1)$ to the end point $\mathbf{r}(t_2)$ is the one that minimizes the action. This variational principle leads to the Euler–Lagrange equations:

$$\frac{\partial L}{\partial r_i} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}_i} \right), \quad r_i = x, y, z. \quad (\text{I.26})$$

An important remark for what follows is the following observation: if we add to a given Lagrangian a total time derivative

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) \longrightarrow L(\mathbf{r}, \dot{\mathbf{r}}, t) + \frac{d}{dt} \Omega(\mathbf{r}, t), \quad (\text{I.27})$$

the corresponding Euler–Lagrange equations are not modified. Indeed

$$\frac{d}{dt} \Omega(\mathbf{r}, t) = \frac{\partial \Omega}{\partial t} + \sum_{j=x,y,z} \dot{r}_j \frac{\partial \Omega}{\partial r_j} \quad (\text{I.28})$$

which leads to adding the same quantity

$$\frac{\partial^2 \Omega}{\partial t \partial r_i} + \sum_{j=x,y,z} \dot{r}_j \frac{\partial^2 \Omega}{\partial r_i \partial r_j} \quad (\text{I.29})$$

to both sides of equation (I.26).

2-3 Lagrangian in a magnetic field

For a free particle, in the absence of a magnetic field, the motion is uniform and rectilinear and corresponds to the choice

$$L_{\text{libre}}(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2} M \dot{\mathbf{r}}^2. \quad (\text{I.30})$$

In the presence of a magnetic field $\mathbf{B}(\mathbf{r})$, is there a Lagrangian that recovers the equation of motion (I.23)? The answer is (multiply) positive: for any vector potential $\mathbf{A}$ associated with the magnetic field $\mathbf{B}$, one can consider

$$L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2} M \dot{\mathbf{r}}^2 + q \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}) \quad (\text{I.31})$$

and check that the Euler–Lagrange equations are indeed identical to (I.23).

The fact that we recover the equation of motion (I.23) guarantees that the choice (I.31), although explicitly involving the gauge, does not lead to predictions that would depend on this choice of gauge. This is of course essential, and one can try to prove this result without going through the explicit Euler–Lagrange equations. Consider a gauge transformation

$$\mathbf{A}(\mathbf{r}) \longrightarrow \mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \nabla \chi(\mathbf{r}), \quad (\text{I.32})$$

corresponding to the change of Lagrangian

$$L(\mathbf{r}, \dot{\mathbf{r}}) \longrightarrow L_\chi(\mathbf{r}, \dot{\mathbf{r}}) = L(\mathbf{r}, \dot{\mathbf{r}}) + q \dot{\mathbf{r}} \cdot \nabla \chi(\mathbf{r}). \quad (\text{I.33})$$

One immediately sees that the two Lagrangians L and L_χ differ by a total time derivative

$$L_\chi(\mathbf{r}, \dot{\mathbf{r}}) = L(\mathbf{r}, \dot{\mathbf{r}}) + \frac{d}{dt} \Omega(\mathbf{r}) \quad \text{with} \quad \Omega(\mathbf{r}) = q \chi(\mathbf{r}), \quad (\text{I.34})$$

and they must indeed lead to the same equations of motion.

2-4 The principle of a gauge theory

One can reverse the thread of the previous reasoning, putting oneself in the place of a physicist who has never heard of the Lorentz force, but is very imaginative. Starting from the Lagrangian of a free particle (I.30), $L = M \dot{\mathbf{r}}^2 / 2$, this physicist can take advantage of the invariance of the equations of motion under the addition of a total time derivative. This invariance guarantees that all Lagrangians of the type

$$L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2} M \dot{\mathbf{r}}^2 + q \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}) \quad (\text{I.35})$$

with the vector field $\mathbf{A}(\mathbf{r})$ such that

$$\mathbf{A}(\mathbf{r}) = \nabla\chi(\mathbf{r}), \quad (\text{I.36})$$

will also describe the motion of a free particle. At this stage, he has merely complicated, somewhat gratuitously, the formalism at his disposal, but he can then ask himself the following question: what type of physical system will he obtain if he generalizes the problem by considering Lagrangians of the type (I.35), but ignoring the constraint (I.36)? The answer is immediate given what precedes: he will invent/discover orbital magnetism, i.e., the motion of a particle of charge q in a magnetic field, with $\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$.

2-5 Hamiltonian in a magnetic field

The passage from the Lagrangian formalism to the Hamiltonian formalism is done by first defining the canonical momenta

$$p_i = \frac{\partial L}{\partial \dot{r}_i}, \quad (\text{I.37})$$

and then considering the Legendre transformation

$$H(\mathbf{r}, \mathbf{p}, t) = \mathbf{p} \cdot \dot{\mathbf{r}} - L(\mathbf{r}, \dot{\mathbf{r}}, t), \quad (\text{I.38})$$

where the velocity $\dot{\mathbf{r}}$ is assumed to be expressed as a function of the momentum $\mathbf{p}$ by inverting (I.37). The motion of the particle(s) is then determined by Hamilton's equations

$$\dot{r}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial r_i}. \quad (\text{I.39})$$

For the Lagrangian of a charged particle in a magnetic field proposed in (I.31), the conjugate momentum is

$$\mathbf{p} = \nabla_{\dot{\mathbf{r}}} L(\mathbf{r}, \dot{\mathbf{r}}) = M\dot{\mathbf{r}} + q\mathbf{A}(\mathbf{r}) \quad (\text{I.40})$$

and the Hamiltonian is thus written, after rearranging terms:

$$H(\mathbf{r}, \mathbf{p}) = \frac{(\mathbf{p} - q\mathbf{A}(\mathbf{r}))^2}{2M}. \quad (\text{I.41})$$

One can check that Hamilton's equations (I.39) then indeed give back the fundamental equation of dynamics (I.23) from which we started.

One immediately sees that the *canonical momentum* $\mathbf{p}$ given in (I.40), also called the *momentum* or *generalized momentum*, is not a physical quantity independent of the gauge. This momentum is in particular different from the *kinetic momentum*

$$\boldsymbol{\pi} = M\dot{\mathbf{r}}, \quad (\text{I.42})$$

since

$$\mathbf{p} = \boldsymbol{\pi} + q\mathbf{A}(\mathbf{r}). \quad (\text{I.43})$$

This distinction is the price to pay for obtaining a velocity-dependent force within the Hamiltonian formalism.

3 Quantum particle in a magnetic field

3-1 Canonical quantization

The value of the classical Hamiltonian formalism is that it lends itself well to the transition to quantum mechanics, by way of the rule of canonical quantization in which the position $\mathbf{r}$ and the momentum $\mathbf{p}$ become operators $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$ obeying

$$[\hat{r}_j, \hat{r}_k] = 0, \quad [\hat{r}_j, \hat{p}_k] = i\hbar \delta_{j,k}, \quad [\hat{p}_j, \hat{p}_k] = 0. \quad (\text{I.44})$$

Let us begin by recalling how these quantization relations impose a very particular form for the action of the position operator $\hat{\mathbf{r}}$ and momentum operator $\hat{\mathbf{p}}$ on wave functions (Dirac 1958). At this stage, we assume nothing about the presence or absence of a magnetic field.

Consider a point particle whose state is described by a complex wave function $\psi(\mathbf{r})$, giving the probability density $\mathcal{P}(\mathbf{r})$ of presence at point $\mathbf{r}$:

$$\mathcal{P}(\mathbf{r}) = |\psi(\mathbf{r})|^2. \quad (\text{I.45})$$

The task is to determine the action of the operators $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$ on this wave function, given the relations (I.44).

We posit by convention that the action of the position operator $\hat{r}$ on $\psi(\mathbf{r})$ is multiplication by the variable $\mathbf{r}$ itself:

$$\hat{r}[\psi(\mathbf{r})] = \mathbf{r} \psi(\mathbf{r}), \quad (\text{I.46})$$

which of course satisfies the first commutation relation $[\hat{r}_j, \hat{r}_k] = 0$ of (I.44). The second commutation relation, which relates the components of $\hat{r}$ and $\hat{p}$, then implies that

$$\hat{r}_j \hat{p}_k [\psi(\mathbf{r})] = \hat{p}_k [r_j \psi(\mathbf{r})] + i\hbar \delta_{j,k} \psi(\mathbf{r}), \quad (\text{I.47})$$

whose solution is

$$\hat{p} = -i\hbar \nabla + \mathbf{X}(\mathbf{r}), \quad (\text{I.48})$$

where $\mathbf{X}(\mathbf{r})$ is, at this stage, an arbitrary vector field. Finally, the third commutation relation $[\hat{p}_j, \hat{p}_k] = 0$ imposes a constraint on the vector field $\mathbf{X}$, with

$$\frac{\partial X_j}{\partial x_k} = \frac{\partial X_k}{\partial x_j} \Rightarrow \nabla \times \mathbf{X} = 0 \Rightarrow \mathbf{X} = \nabla \Omega(\mathbf{r}), \quad (\text{I.49})$$

where Ω is an arbitrary scalar function of $\mathbf{r}$. The rule of canonical quantization thus amounts to imposing the action of the operators $\mathbf{r}$ and $\mathbf{p}$ on a wave function:

$$\hat{r}[\psi(\mathbf{r})] = \mathbf{r} \psi(\mathbf{r}), \quad (\text{I.50})$$

$$\hat{p}[\psi(\mathbf{r})] = -i\hbar \nabla \psi(\mathbf{r}) + (\nabla \Omega(\mathbf{r})) \psi(\mathbf{r}). \quad (\text{I.51})$$

3-2 Gauge transformation for quantum mechanics

The choice of the function Ω in (I.51) is arbitrary, but has no physical consequence (Dirac 1958). Indeed, for a given choice of Ω , one can introduce the unitary transformation, called the *gauge transformation* (and not related at this stage to an electromagnetic gauge transformation)

$$\psi(\mathbf{r}) \longrightarrow \hat{T} \psi(\mathbf{r}), \quad \text{with} \quad \hat{T} = \exp[i\Omega(\hat{\mathbf{r}})/\hbar], \quad (\text{I.52})$$

which does not modify the probability density $\mathcal{P}(\mathbf{r})$. Under this transformation, the operators $\hat{r}$ and $\hat{p}$ become

$$\hat{r} \longrightarrow \hat{\hat{r}} = \hat{T} \hat{r} \hat{T}^\dagger = \hat{r}, \quad (\text{I.53})$$

$$\hat{p} = -i\hbar \nabla + \nabla \Omega(\mathbf{r}) \longrightarrow \hat{\hat{p}} = \hat{T} \hat{p} \hat{T}^\dagger = -i\hbar \nabla, \quad (\text{I.54})$$

which eliminates the function Ω . In everything that follows (whether a magnetic field is present or not), we will assume that this transformation has been performed, and we will thus adopt the usual convention

$$\hat{r}[\psi(\mathbf{r})] = \mathbf{r} \psi(\mathbf{r}), \quad \hat{p}[\psi(\mathbf{r})] = -i\hbar \nabla \psi(\mathbf{r}). \quad (\text{I.55})$$

3-3 Quantum Hamiltonian

Once the operators $\hat{r}$ and $\hat{p}$ have been constructed, one can deduce from them the operator associated with any physical quantity depending on position and momentum, in particular the Hamiltonian of a particle of electric charge q in the presence of a static magnetic field

$$\hat{H} = \frac{(\hat{p} - q\mathbf{A}(\hat{\mathbf{r}}))^2}{2M} = \frac{(-i\hbar \nabla - q\mathbf{A}(\mathbf{r}))^2}{2M}. \quad (\text{I.56})$$

This is the form we will use throughout this course, and which we will in particular try to simulate for electrically neutral particles.

Like its classical counterpart, this expression depends explicitly on the gauge chosen to determine the vector potential. Consider a wave function $\psi(\mathbf{r}, t)$ solving the Schrödinger equation

$$i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = \frac{(-i\hbar \nabla - q\mathbf{A}(\mathbf{r}))^2}{2M} \psi(\mathbf{r}, t). \quad (\text{I.57})$$

An electromagnetic gauge transformation

$$\mathbf{A}(\mathbf{r}) \longrightarrow \mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \nabla \chi(\mathbf{r}), \quad (\text{I.58})$$

will modify the Hamiltonian, and $\psi(\mathbf{r}, t)$ will no longer be a solution of the Schrödinger equation written with the vector potential $\mathbf{A}'$. On the other hand, if we accompany this electromagnetic gauge transformation with the quantum gauge transformation similar to (I.52)

$$\psi(\mathbf{r}, t) \longrightarrow \psi'(\mathbf{r}, t) = \hat{T} \psi(\mathbf{r}, t), \quad \text{with} \quad \hat{T} = \exp[iq\chi(\hat{\mathbf{r}})/\hbar], \quad (\text{I.59})$$

then the wave function $\psi'(\mathbf{r}, t)$ obtained in this way is indeed a solution of the Schrödinger equation for the vector potential $\mathbf{A}'$:

$$i\hbar \frac{\partial \psi'}{\partial t} = \frac{(-i\hbar \nabla - q\mathbf{A}'(\mathbf{r}))^2}{2M} \psi'(\mathbf{r}, t). \quad (\text{I.60})$$

This shows the deep connection between the gauge symmetry of the Schrödinger equation, expressed by the unitary transformation (I.52), and the gauge invariance of Maxwell’s equations considered here at the level of magnetostatics (Cohen-Tannoudji, Diu, et al. (1973), chapter 3, complement H_{III}).

3-4 Quantum + electromagnetic gauge invariance

One can reverse the previous approach to arrive at a formulation of quantum mechanics in which magnetism (or electromagnetism) appears naturally via the addition of a local gauge symmetry condition. We have already sketched this reasoning in the classical setting (§ 2-4), and we now transpose it to the quantum setting.

Consider a free particle, of mass M , of charge q , and with wave function $\psi(\mathbf{r}, t)$ solving the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi \quad (\text{I.61})$$

without giving the form of $\hat{H}$ at this stage. Let us impose only the following additional condition, corresponding to local gauge symmetry⁴:

The physics must remain the same if we perform the gauge transformation

$$\psi \rightarrow \tilde{\psi} = \hat{T} \psi \quad \text{with} \quad \hat{T} = \exp[iq\chi(\hat{\mathbf{r}})/\hbar]. \quad (\text{I.62})$$

Clearly, the choice

$$\hat{H} = -\frac{\hbar^2}{2m} \nabla^2 + V(\hat{\mathbf{r}}) \quad (\text{I.63})$$

does not work, since the equation satisfied by $\tilde{\psi}$ will not involve a Hamiltonian keeping the same structure. On the other hand, it is clear from the above that the generic choice

$$\hat{H} = \frac{(-i\hbar \nabla - q\mathbf{A}(\mathbf{r}))^2}{2m} + V(\mathbf{r}) \quad (\text{I.64})$$

⁴We do not perform here an equivalent transformation on the observables, since otherwise the unitarity of quantum mechanics would make this transformation trivial.

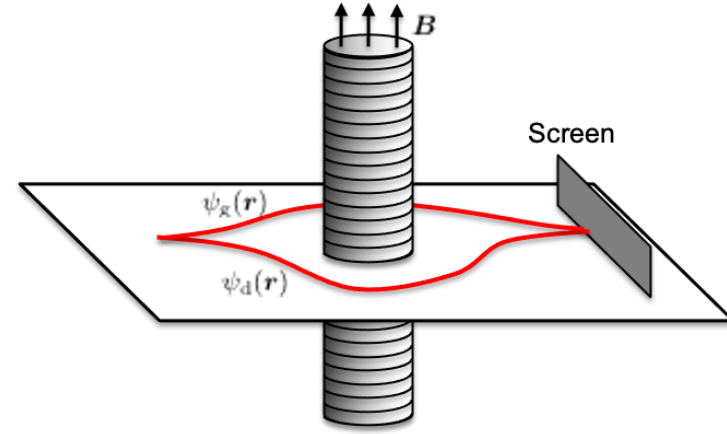


Figure I.2. Geometry considered by Aharonov and Bohm.

associated with the modification of $\mathbf{A}$

$$\mathbf{A}(\mathbf{r}) \rightarrow \mathbf{A}(\mathbf{r}) + \nabla \chi(\mathbf{r}) \quad (\text{I.65})$$

works.

To ensure this local gauge symmetry, it is sufficient to restrict ourselves to vector potentials $\mathbf{A}(\mathbf{r})$ equal to the gradient of a scalar function $\Omega(\mathbf{r})$. But the imaginative physicist mentioned above may wonder what happens if this choice of “generalized” Hamiltonian is extended to an arbitrary vector potential $\mathbf{A}$ (not necessarily curl-free). The answer is, again, simple: our physicist will discover quantum orbital magnetism!

4 The Aharonov–Bohm effect

In their famous 1959 article, entitled *Significance of Electromagnetic Potentials in the Quantum Theory*⁵, Aharonov and Bohm proposed a thought experi-

⁵The point of interest to us here is the fact that the fields $\mathbf{E}$ and $\mathbf{B}$ are both zero in the region accessible to the particles, thereby revealing the importance of gauge potentials. Vari-

ment that reveals a remarkable fact⁶ : one can detect the presence of a magnetic field through measurements made on quantum particles, even if these particles are never found in regions where $\mathbf{B}(\mathbf{r}) \neq 0$ (Aharonov & Bohm 1959).

The geometry proposed by Aharonov and Bohm uses the infinite solenoid already considered in (1-3), placed in the middle of a two-path interferometer located in the xy plane. The solenoid itself is surrounded by a potential barrier so that the particles do not enter it; the interference observed on the detection screen thus results from the superposition of the two waves $\psi_g(\mathbf{r})$ and $\psi_d(\mathbf{r})$, having passed respectively to the left and to the right of the solenoid. The crucial result is that the interference pattern is not the same depending on whether a current flows in the solenoid or not, even though the magnetic field in the region accessible to the particles (outside the solenoid) is zero in both cases.

4-1 The argument of Aharonov and Bohm

Let $\psi_g^{(0)}(\mathbf{r})$ and $\psi_d^{(0)}(\mathbf{r})$ denote the two matter waves in the absence of current in the solenoid, corresponding to the gauge choice $\mathbf{A} = 0$ throughout space. When a current flows in the solenoid, the vector potential $\mathbf{A}(\mathbf{r})$ is no longer zero outside the solenoid, but the magnetic field $\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$ remains zero in the region accessible to the particles. We thus have, in this region,

$$\nabla \times \mathbf{A}(\mathbf{r}) = 0. \quad (\text{I.66})$$

How can this differential equation be integrated? In a “hole-free” (simply connected) region, the solution is simple. There exists a scalar function $\chi(\mathbf{r})$ such that

$$\mathbf{A}(\vec{r}) = \nabla \chi(\mathbf{r}). \quad (\text{I.67})$$

ants of this effect relax some of these constraints, such as the Aharonov–Casher effect, where the presence of an electrostatic field modifies, by a geometric phase, the wave function of a magnetic dipole (Aharonov & Casher 1984).

⁶We use here the standard terminology of the “Aharonov–Bohm effect,” although Ehrenberg and Siday, in a 1949 article, proposed a similar thought experiment and reached the same conclusion: *The irremovable anisotropy of the field-free region as a whole emphasizes the fact that the electron-optical refractive index contains the vector potential and not the magnetic field strength.* (Ehrenberg & Siday 1949)

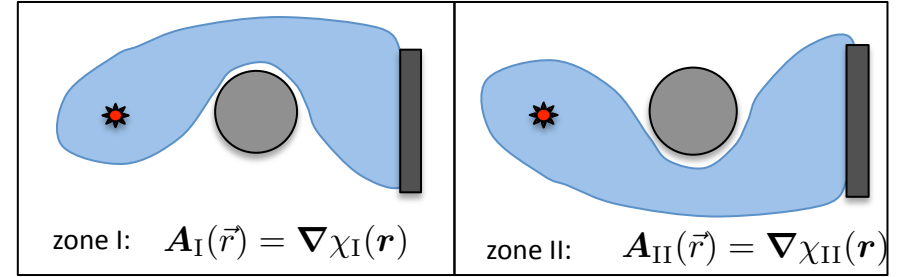


Figure I.3. Two zones (patches) over which the vector potential in the presence of current can be written as $\mathbf{A}(\vec{r}) = \nabla \chi(\mathbf{r})$. Note that the function χ is not the same for the two zones!

Indeed, in a hole-free region, we have for any closed circuit $\mathcal{C}$

$$\nabla \times \mathbf{A}(\mathbf{r}) = 0 \quad \Rightarrow \quad \oint_{\mathcal{C}} \mathbf{A}(\mathbf{r}) \cdot d\mathbf{r} = 0, \quad (\text{I.68})$$

which allows us to unambiguously define

$$\chi(\mathbf{r}) = \chi(\mathbf{0}) + \int_0^r \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}', \quad (\text{I.69})$$

where we have chosen an origin denoted $\mathbf{0}$ and fixed the value $\chi(\mathbf{0})$ of χ at this point.

On the other hand, this reasoning is not valid for the Aharonov–Bohm geometry, because the region accessible to the particles is not simply connected; indeed, the region occupied by the solenoid has been removed from the xy plane. However, one can consider two regions of space denoted I and II in figure I.3, corresponding to the “trajectories” passing to the left and to the right of the solenoid. For each of these regions, one can solve (I.66) in the form (I.67) and write:

$$\text{zone I: } \mathbf{A}_I(\vec{r}) = \nabla \chi_I(\mathbf{r}), \quad \text{zone II: } \mathbf{A}_{II}(\vec{r}) = \nabla \chi_{II}(\mathbf{r}). \quad (\text{I.70})$$

Let us then consider the wave ψ_g , which propagates only in region I. For this wave, switching on the current in the solenoid amounts to making

the change

$$\mathbf{A}(\mathbf{r}) = 0 \longrightarrow \mathbf{A}_I(\mathbf{r}) = \nabla \chi_I(\mathbf{r}), \quad (\text{I.71})$$

which corresponds to a simple gauge transformation. This must be accompanied by a modification of the wave function [c.f. Eq. (I.65)]

$$\psi_g^{(0)}(\mathbf{r}) \longrightarrow \psi_g(\mathbf{r}) = \exp[iq\chi_I(\mathbf{r})/\hbar] \psi_g^{(0)}(\mathbf{r}). \quad (\text{I.72})$$

In the same way, switching on the current in the solenoid amounts to changing the wave ψ_d as follows:

$$\psi_d^{(0)}(\mathbf{r}) \longrightarrow \psi_d(\mathbf{r}) = \exp[iq\chi_{II}(\mathbf{r})/\hbar] \psi_d^{(0)}(\mathbf{r}). \quad (\text{I.73})$$

Let us adopt the convention $\chi_I(\mathbf{0}) = \chi_{II}(\mathbf{0}) = 0$. The interference of the matter waves at a point $\mathbf{r}$ on the screen involves

$$\psi_d^*(\mathbf{r}) \psi_g(\mathbf{r}) = \exp([iq(\chi_I(\mathbf{r}) - \chi_{II}(\mathbf{r}))/\hbar]) \psi_d^{(0)*}(\mathbf{r}) \psi_g^{(0)}(\mathbf{r}), \quad (\text{I.74})$$

and the argument of the prefactor can be rewritten

$$\Phi = \chi_I(\mathbf{r}) - \chi_{II}(\mathbf{r}) = \int_{0, C_I}^{\mathbf{r}} \mathbf{A}(\mathbf{r}) \cdot d\mathbf{r} - \int_{0, C_{II}}^{\mathbf{r}} \mathbf{A}(\mathbf{r}) \cdot d\mathbf{r}, \quad (\text{I.75})$$

where C_I and C_{II} are arbitrary paths going from 0 to $\mathbf{r}$, lying respectively within regions I and II. Let us introduce a closed path $\mathcal{C}$ going from 0 to $\mathbf{r}$ through region I and returning from $\mathbf{r}$ to 0 through region II; this path thus goes once around the solenoid, and we have

$$\Phi = \int_{\mathcal{C}} \mathbf{A}(\mathbf{r}) \cdot d\mathbf{r} = \iint_{\mathcal{C}} \mathbf{u}_z \cdot \mathbf{B}(\mathbf{r}') d^2r' \quad (\text{I.76})$$

which shows that Φ is nothing other than the flux of $\mathbf{B}$ through the solenoid. The relative phase of the two matter waves arriving at the screen therefore involves this flux, even though the particles never enter the region where the field is present! Note that the phase $q\Phi/\hbar$ can be called *topological*, since it remains the same when the contour $\mathcal{C}$ is continuously deformed, provided it does not cross the solenoid.

4-2 Argument based on the path integral

In this paragraph, we will recover the result of Aharonov and Bohm by a slightly different method, one that does not appeal to the two vector potentials $\mathbf{A}_I$ and $\mathbf{A}_{II}$, but instead uses the interaction Lagrangian between the charged particle and the field via the path-integral formalism (Feynman & Hibbs 1965).

In this formalism, one computes the propagator giving the probability amplitude for a particle starting from $\mathbf{0}$ at time t_1 to reach the point $\mathbf{r}$ on the screen at time t_2 . This propagator is given by the sum of the $e^{iS_\Gamma/\hbar}$, where S_Γ is the action associated with a given path Γ going from $(\mathbf{0}, t_1)$ to $(\mathbf{r}, t_2)$

$$K(\mathbf{0}, t_1; \mathbf{r}, t_2) \propto \sum_{\text{chemins } \Gamma} \exp(iS_\Gamma/\hbar), \quad S_\Gamma = \int_{t_1}^{t_2} L[\mathbf{r}(t), \dot{\mathbf{r}}(t), t] dt. \quad (\text{I.77})$$

The Lagrangians in the absence and in the presence of the magnetic field are, respectively,

$$L^{(0)}(\mathbf{r}, \dot{\mathbf{r}}) \quad \text{and} \quad L^{(0)}(\mathbf{r}, \dot{\mathbf{r}}) + q \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}), \quad (\text{I.78})$$

where we have included in $L^{(0)}$ the potential $V(\mathbf{r})$ used to guide the particle on either side of the solenoid. One then notices that the additional term in the action of a given path

$$q \int_{t_1}^{t_2} \dot{\mathbf{r}}(t) \cdot \mathbf{A}[\mathbf{r}(t)] dt = q \int_{0, \Gamma}^{\mathbf{r}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}' \quad (\text{I.79})$$

takes the same value $\delta S_I(\mathbf{r})$ for all paths Γ_I restricted to zone I, and another value $\delta S_{II}(\mathbf{r})$ for all paths Γ_{II} restricted to zone II [c.f. (I.68) and (I.69)].

This remarkable result makes it possible to evaluate the change in the interference signal caused by switching on the current in the solenoid, even though it is *a priori* very difficult to compute the propagator K in the absence of current. To this end, let us neglect the contribution of “exotic” paths that wind one or more full times around the solenoid on their way from the source to the screen. In the absence of current, the propagator from $(\mathbf{0}, t_1)$ to $(\mathbf{r}, t_2)$ can then be written as the sum $K(\mathbf{r}) \approx K_I(\mathbf{r}) + K_{II}(\mathbf{r})$ of the contributions passing to the left or to the right of the solenoid. In the

presence of the current, this propagator is modified and becomes

$$K(\mathbf{r}) \approx K_{\text{I}}(\mathbf{r}) + K_{\text{II}}(\mathbf{r}) \longrightarrow K_{\text{I}}(\mathbf{r}) e^{i\delta S_{\text{I}}(\mathbf{r})/\hbar} + K_{\text{II}}(\mathbf{r}) e^{i\delta S_{\text{II}}(\mathbf{r})/\hbar} \quad (\text{I.80})$$

The change in the relative phase between K_{I} and K_{II} due to the current is therefore

$$\begin{aligned} \frac{1}{\hbar}(\delta S_{\text{I}}(\mathbf{r}) - \delta S_{\text{II}}(\mathbf{r})) &= \frac{q}{\hbar} \left(\int_{0,\text{I}}^{\mathbf{r}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}' - \int_{0,\text{II}}^{\mathbf{r}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}' \right) \\ &= \frac{q}{\hbar} \oint_{\mathcal{C}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}' = q\Phi/\hbar, \end{aligned} \quad (\text{I.81})$$

where $\mathcal{C}$ is a closed circuit encircling the solenoid. This indeed recovers the result of the previous paragraph.

We have thus developed two lines of reasoning to find the phase shift in an Aharonov–Bohm-type experiment:

1. One can consider domains of space (*patches*) over which vector potentials $A_{\text{I}}(\mathbf{r})$, $A_{\text{II}}(\mathbf{r})$, ..., are defined, and glue these pieces together (i) at the point $\mathbf{0}$ by setting $\chi_{\text{I}}(\mathbf{0}) = \chi_{\text{II}}(\mathbf{0}) = 0$ and (ii) at the point $\mathbf{r}$, the sought-after phase shift appearing at this second gluing.
2. One can take advantage of the fact that, for this geometry, the vector potential is known throughout space, and one then uses the particle-field interaction Lagrangian to compute the phase shift.

We will see a little further on, in the paragraph devoted to magnetic monopoles, that one does not always have a choice between these two possibilities, and that the first method, that of gluing together two or more zones, is sometimes unavoidable.

4-3 Experimental demonstration

As soon as the article by Aharonov and Bohm was published, experimental tests of this effect were carried out [see for example Chambers (1960)]. The results of these experiments were the subject of debate, stemming from the difficulty of quantitatively estimating edge effects: one must ensure that the Lorentz force due to stray magnetic fields indeed plays a negligible

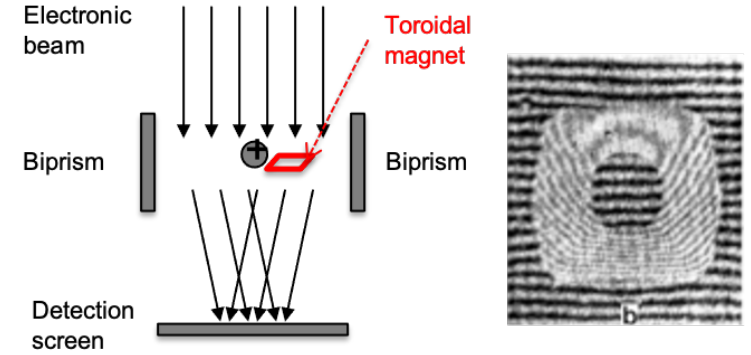


Figure I.4. Left: Fresnel (electrostatic) biprism experiment carried out with an electron beam. A toroidal magnet is placed in one of the two paths of the interferometer. Right: interference pattern obtained with the toroidal magnet. The shift of the fringes between the particles that passed inside the torus and those that passed outside is the signature of the Aharonov–Bohm effect (figure taken from Tonomura, Matsuda, et al. 1982).

role. In a series of experiments carried out in the Hitachi laboratories between 1982 and 1986, A. Tonomura and his team adopted a geometry that made it possible to properly control these edge effects (Tonomura, Matsuda, et al. 1982; Tonomura, Osakabe, et al. 1986). Their solenoid is not straight, as in the Aharonov and Bohm thought experiment, but toroidal, which makes it possible to minimize stray fields.

Tonomura’s experiments use a collimated electron beam, which is directed onto a detection screen after passing through a Fresnel biprism (Tonomura 1987). A very schematic version is shown in figure I.4. In the absence of any perturbation, this leads to straight interference fringes. In the 1982 experiment, Tonomura *et al.* placed on one of the two paths a square-shaped toroidal Permalloy magnet, with outer sides of 3 microns and inner sides of 1 micron. Given the magnetic properties of Permalloy, the expected phase shift between a trajectory passing outside the torus and another passing inside corresponds to a path difference of 6λ , where the wavelength λ of the electrons is $\lambda = 0.03$ angstroms for an electron accelerating voltage of 150 kV. The experiment quantitatively confirms this phase

shift, with a measured phase difference of 5.5λ (figure I.4).

This experiment was refined in 1985 by depositing another material (niobium) on the surface of the Permalloy torus. When the temperature is lowered sufficiently, the niobium becomes superconducting, which has the effect of screening the magnetic field of the Permalloy. The quantization of the flux in a superconducting loop implies that the phase shift between trajectories inside and outside the torus can only take the values 0 or π (modulo 2π). This magnetic-flux quantization effect was indeed demonstrated in the interferograms produced by this Fresnel biprism (Tonomura, Osakabe, et al. 1986).

5 Magnetic monopole and quantum physics

We conclude this chapter with a question raised by Dirac, concerning the consequence of the possible existence of magnetic monopoles. Let us recall that, to this day, no monopole has been observed; however, Dirac noticed that it would suffice for a single monopole to exist for the electric charge to be quantized, within the quantum formalism (which does seem to be the case experimentally) (Dirac 1931; Dirac 1948). In what follows, we will briefly sketch Dirac's reasoning, following quite closely the treatment of Sakurai & Napolitano (2011). We will conclude with the description of a recent experiment, simulating the creation of a magnetic monopole in a spinor Bose–Einstein condensate.

5-1 Quantization of the electric charge

Suppose that a magnetic monopole of magnetic charge q_m is present and positioned at the origin of coordinates, creating the field

$$\mathbf{B}_m = \frac{q_m}{4\pi r^2} \mathbf{u}_r. \quad (\text{I.82})$$

We are going to be interested in the wave function of another particle of electric charge q_e moving around the monopole, an electron for example.

Let us first note that there is no vector potential $\mathbf{A}(\mathbf{r})$ defined everywhere in space except at the monopole point that would recover $\mathbf{B}_m$ via

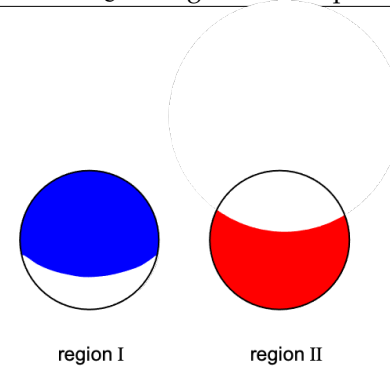


Figure I.5. Graphical representation of the regions I and II, defined here by their polar angles in spherical coordinates.

$\mathbf{B}_m = \nabla \times \mathbf{A}$. Indeed, for a sphere S centered on the origin, we find

$$\iint_S \mathbf{n} \cdot \mathbf{B} \, d^2r = q_m, \quad (\text{I.83})$$

whereas the flux of a curl through a closed surface is zero.

One might legitimately fear that our Lagrangian or Hamiltonian formalism, based on the existence of the vector potential, becomes unworkable. However, one can “rescue” this formalism by taking advantage of the fact that one does not really need a single version of the vector potential over the whole of space. One can continue to use this formalism if one manages (i) to tile space with zones where a given type of vector potential is well defined, and (ii) to glue these zones together. This is in fact a technique we have already presented to study the Aharonov–Bohm effect. Here, we will cut space into two zones denoted I and II, and define a vector potential $A_I(\mathbf{r})$ and $A_{II}(\mathbf{r})$ on each zone. It is from the gluing of the wave functions associated with $A_I(\mathbf{r})$ and $A_{II}(\mathbf{r})$ that the quantization condition for the charge q_e will emerge.

A possible choice for the vector potentials $A_I(\mathbf{r})$ and $A_{II}(\mathbf{r})$ has already been given above. Let us recall it here:

$$A_I(\mathbf{r}) = \frac{q_m}{4\pi} \frac{1 - \cos \theta}{r \sin \theta} \mathbf{u}_\varphi, \quad A_{II}(\mathbf{r}) = -\frac{q_m}{4\pi} \frac{1 + \cos \theta}{r \sin \theta} \mathbf{u}_\varphi. \quad (\text{I.84})$$

We know that A_I must not be used near the half-axis $z < 0$, since it gives rise there to a Dirac string going from $z = 0$ to $z = -\infty$. Likewise A_{II}

must not be used near the half-axis $z > 0$, for the same reason (a string extending from $z = 0$ to $z = +\infty$). To fix ideas, let us therefore take the following regions (see figure I.5)

- Region I is defined as the set of points $\mathbf{r}$ in space, with spherical coordinates $0 \leq \theta < 3\pi/4$ and $r \neq 0$.
- Region II is defined as the set of points $\mathbf{r}$ in space, with spherical coordinates $\pi/4 < \theta \leq \pi$ and $r \neq 0$.

In the overlap zone $I \cap II$, i.e., the zone around the equator corresponding to $\pi/4 < \theta < 3\pi/4$, the two vector potentials describe the same magnetic field and are therefore related by a gauge transformation⁷. In this case, the gauge transformation taking $A_I(\mathbf{r})$ to $A_{II}(\mathbf{r})$ is simple:

$$A_{II}(\mathbf{r}) - A_I(\mathbf{r}) = -\frac{q_m}{2\pi} \frac{1}{r \sin \theta} \mathbf{u}_\varphi = \nabla \chi(\mathbf{r}) \quad (\text{I.85})$$

$$\text{with } \chi(\mathbf{r}) = \chi_0 - \frac{q_m}{2\pi} \varphi, \quad \mathbf{r} \in I \cap II, \quad (\text{I.86})$$

Let us now consider a quantum particle, and let ψ_I and ψ_{II} respectively denote the wave functions corresponding to the gauge choice $A_I(\mathbf{r})$ and $A_{II}(\mathbf{r})$. In the zone $I \cap II$, where one passes from one gauge to the other via (I.86), the two wave functions ψ_I and ψ_{II} must also be related by this gauge transformation

$$\psi_{II}(\mathbf{r}) = e^{iq_e \chi(\mathbf{r})/\hbar} \psi_I(\mathbf{r}), \quad \mathbf{r} \in I \cap II, \quad (\text{I.87})$$

It is then simple to conclude: according to the postulates of quantum mechanics, the wave functions $\psi_I(\mathbf{r})$ and $\psi_{II}(\mathbf{r})$ must be single-valued⁸. This imposes in particular that they do not change value under the change of spherical coordinates $(r, \theta, \varphi) \rightarrow (r, \theta, \varphi + 2\pi)$:

$$\psi_{I,II}(r, \theta, \varphi + 2\pi) = \psi_{I,II}(r, \theta, \varphi) \quad (\text{I.88})$$

⁷This is not true for the whole of space, since $A_I(\mathbf{r})$ and $A_{II}(\mathbf{r})$ correspond to different fields on the z axis.

⁸Recall that it is this postulate that makes it possible to prove that orbital angular momenta are integers, and not half-integers, as the general theory of angular momentum might have suggested.

This imposes that $e^{iq_e \chi(\mathbf{r})/\hbar}$ also does not change when $\varphi \rightarrow \varphi + 2\pi$, i.e.,

$$\exp\left(i \frac{q_e q_m}{\hbar}\right) = 1 \quad \Rightarrow \quad q_e q_m = n h, \quad (\text{I.89})$$

where n is an arbitrary integer and $h = 2\pi\hbar$. This is the announced quantization⁹ !

A different approach to the problem can be found in Jackson's book *Classical Electrodynamics*¹⁰, where one first computes the angular momentum of the electromagnetic field created by the superposition of the electric charge q_e and the magnetic charge q_m , and then posits that this angular momentum must be quantized in multiples of $\hbar/2$.

5-2 Simulation of a monopole in a condensate

In recent years, order-parameter structures reminiscent of a magnetic monopole have been demonstrated in several condensed-matter systems. This is, of course, only an analogy: these monopoles do not generate a real magnetic field and do not imply, in any way, the quantization of electric charge foreseen by Dirac. Nevertheless, their study is interesting because it reveals the characteristic topology of monopoles, together with the Dirac string attached to them. The first of these experiments were carried out in *spin ice*-type materials (Castelnovo, Moessner, et al. 2008; Morris, Tennant, et al. 2009; Fennell, Deen, et al. 2009). We will describe here a very recent experiment carried out in the USA (Amherst College) by D. S. Hall and his team on cold atoms. These researchers demonstrated a "spin texture" in a Bose-Einstein condensate, which leads, for the wave function of this condensate, to a Schrödinger-type evolution equation in the presence of a monopole (Ray, Ruokokoski, et al. 2014).

We will not develop here the whole formalism needed to quantitatively analyze the experiment of Ray, Ruokokoski, et al. (2014), but we indicate the few essential elements needed to understand how an artificial monopole magnetic field can emerge in the context of a spinor condensate.

⁹It is interesting to study the Aharonov-Bohm effect for the half-solenoids capable of creating the vector potentials A_I and A_{II} , if the quantization (I.89) holds.

¹⁰Our notation connects to Jackson's by setting $q_m = 4\pi g$, $q_e = e/c$.

These elements are, moreover, closely related to developments we will encounter later in this course, when we study the generation of artificial fields from the Berry phase.

To present the theoretical context, we will draw on a pioneering article by Ho & Shenoy (1996), who showed how the nonzero spin of the alkali atoms used in gaseous condensate experiments makes it possible to envisage physics richer than that of a scalar condensate, such as superfluid helium. Consider an atomic condensate formed of spin-1 atoms. We will assume that all the atoms are in the same state, written in the form of a macroscopic wave function

$$\Psi(\mathbf{r}) = \psi(\mathbf{r}) \begin{pmatrix} \zeta_{+1}(\mathbf{r}) \\ \zeta_0(\mathbf{r}) \\ \zeta_{-1}(\mathbf{r}) \end{pmatrix} \quad \text{with} \quad \sum_{m=-1}^{+1} |\zeta_m(\mathbf{r})|^2 = 1. \quad (\text{I.90})$$

The scalar wave function $\psi(\mathbf{r})$ represents the probability amplitude for finding an atom at point $\mathbf{r}$, independently of its spin state; the normalized spin¹¹ $|\zeta\rangle$ gives the distribution of the population among the three spin states. Note that the expression (I.90) is not unique, since one can multiply ψ by $e^{i\chi(\mathbf{r})}$ and $|\zeta\rangle$ by $e^{-i\chi(\mathbf{r})}$, leaving the total state Ψ unchanged.

Consider the evolution equation for $\Psi(\mathbf{r}, t)$, restricting ourselves to the kinetic energy, which is the relevant term for bringing out the required gauge field. Let us start from

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2M} \Delta \Psi + \dots \quad (\text{I.91})$$

The time derivative of the left-hand side of (I.91) and the Laplacian of the right-hand side give, respectively,

$$\begin{aligned} i\hbar \frac{\partial \Psi}{\partial t} &= i\hbar \frac{\partial \psi}{\partial t} |\zeta\rangle + i\hbar \psi \frac{\partial |\zeta\rangle}{\partial t} \\ \Delta \Psi &= \Delta(\psi |\zeta\rangle) = (\Delta \psi) |\zeta\rangle + 2(\nabla \psi) \cdot (\nabla |\zeta\rangle) + \psi (\Delta |\zeta\rangle). \end{aligned}$$

Let us now multiply (I.91) by the bra (row vector)

$$\langle \zeta | = (\zeta_{+1}^*(\mathbf{r}), \zeta_0^*(\mathbf{r}), \zeta_{-1}^*(\mathbf{r})). \quad (\text{I.92})$$

¹¹Here we denote with a ket the vectors of the Hilbert space of the spin, and in bold type the vectors of Euclidean space R^3 .

Using $\langle \zeta | \zeta \rangle = 1$ [cf. (I.90)], we arrive at an evolution equation for ψ which can be written in the form

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2M} \Delta \psi + i \frac{\hbar}{M} \nabla \psi \cdot \mathcal{A} + \dots \quad \text{with} \quad \mathcal{A}(\mathbf{r}) = i\hbar \langle \zeta | (\nabla |\zeta\rangle), \quad (\text{I.93})$$

where we have omitted terms proportional to ψ itself. We thus see appearing in the evolution equation for ψ a term in $\hat{\mathbf{p}} \cdot \mathcal{A}$, characteristic of a vector potential: it is exactly this type of term that one seeks to create in order to generate an artificial gauge field. This term is accompanied by other, more standard contributions that we have not written down, corresponding to a scalar potential acting on the particles. We will return to these contributions in later lectures via a more general formalism.

At this stage, writing down an equation of motion for ψ may appear somewhat artificial, since it represents only part of the evolution of the degrees of freedom of the state Ψ . In principle, it must therefore be supplemented by an evolution equation for $|\zeta\rangle$, and the dynamics then becomes complex to analyze. There exists, however, a class of problems where this second step is not necessary: it suffices to “freeze” the spin degree of freedom, by preparing the system in a given spin state and ensuring that the atoms will adiabatically follow this state as the external magnetic field applied to the system is varied.

This is what was done in the experiment of Ray, Ruokokoski, et al. (2014). The authors started from a condensate initially in the state $|F = 1, m_z = +1\rangle$ in a homogeneous magnetic field parallel to the z axis. They then added an inhomogeneous component to this field, corresponding to a quadrupole field, to obtain a total field of the form

$$\mathbf{B}(\mathbf{r}, t) = \begin{pmatrix} b'x \\ b'y \\ -2b'z + B_0(t) \end{pmatrix}. \quad (\text{I.94})$$

This field vanishes at the point $x = y = 0, z = B_0/2b'$. By slowly decreasing $B_0(t)$, the authors moved this zero of the magnetic field along a vertical line, arriving from above the condensate to bring it to its center (figure I.6, left). This causes an adiabatic flip of the spins in the upper part of the condensate, where the atoms end up in the state $|F = 1, m_z = -1\rangle$. In the central part, the atoms are predominantly in the state $|F = 1, m_z = 0\rangle$, and they remained essentially in the state $|F = 1, m_z = +1\rangle$ in the

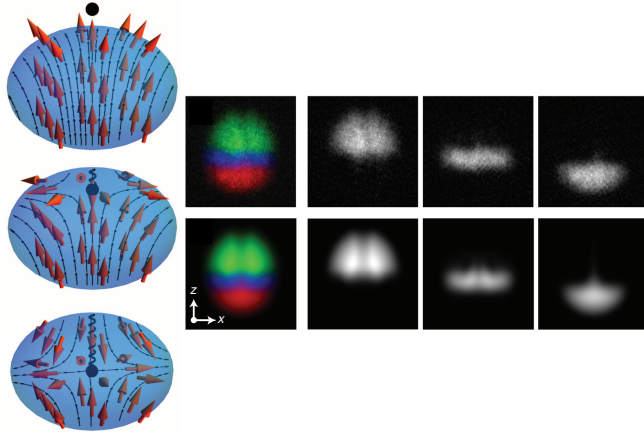


Figure I.6. Left: entry of the magnetic-field zero (the black dot) into a condensate, obtained by varying the field B_0 in equation (I.94). Right: measured (top) and calculated (bottom) spin texture when the field zero is brought to the center of the condensate. Green: $m_z = -1$, blue: $m_z = 0$, red: $m_z = +1$. One can distinguish the furrow created by the passage of the field zero in the $m_z = -1$ and $m_z = 0$ components (Ray, Ruokokoski, et al. 2014).

lower part. We have reproduced on the right of figure I.6 some results of Ray, Ruokokoski, et al. (2014), showing this spatial distribution of the spin components in the condensate once the field zero has been placed at the center of the condensate. One can notably see the furrow created by the passage of the field zero through the upper part of the condensate, within the spin components $m_z = -1$ and $m_z = 0$.

Once the field zero has reached the center of the condensate, one can show that the spin texture $|\zeta\rangle$ of the condensate leads to an artificial vector potential $i\hbar \langle \zeta | (\nabla |\zeta\rangle)$ identical¹² to the one envisaged for creating a monopole:

$$\mathcal{A}(\mathbf{r}) \propto \frac{1 + \cos \theta}{r \sin \theta} \mathbf{u}_\varphi. \quad (\text{I.95})$$

One thus finds a monopole-field structure for $\mathcal{B} = \nabla \times \mathcal{A}$ (figure I.7). Physically, these vector fields $\mathcal{A}(\mathbf{r})$ and $\mathcal{B}(\mathbf{r})$ are related to the superfluid

¹²after a rescaling of the z coordinate.

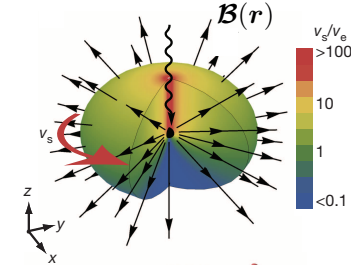


Figure I.7. Structure of the artificial magnetic field $\mathcal{B} = \nabla \times \mathcal{A}$, where the artificial vector potential $\mathcal{A}$ is deduced from the spin texture via (I.93) (Ray, Ruokokoski, et al. 2014).

velocity field and its vorticity (Ho & Shenoy 1996). The evolution of the scalar function ψ is, in principle, the same as that of a quantum electric charge placed in the field of a magnetic monopole. In practice, the long-term dynamics is complicated by the fact that the spin of the atoms may fail to adiabatically follow the motion of their center of mass in regions of very weak field.

Chapter II

Landau levels

This chapter is devoted to a problem essential to the study of magnetism, namely the motion of a charged particle in a uniform magnetic field. After briefly recalling the results obtained in classical physics (Newtonian dynamics), we will move on to the quantum case, which we will treat in several ways corresponding to different choices of gauge. In particular, we will bring out the notion of Landau levels for the eigenspaces of the Hamiltonian; we will see that these levels have a macroscopic degeneracy, which grows linearly with the area of the sample. We will be particularly interested in the lowest Landau level (LLL for *Lowest Landau Level*), for which we will give two explicit expressions for a basis of eigenstates.

The last part of the chapter will be devoted to a few particularly important physical effects that will appear as a direct consequence of this quantization into Landau levels. In particular, we will bring out the notion of edge state, which plays a central role in the physics of the quantum Hall effect. We will illustrate this notion by describing an interferometry experiment between two electronic circuits carried out by Ji, Chung, et al. (2003). Finally, we will present a proposed experiment making it possible to demonstrate these edge states for cold atoms confined in an optical lattice (Goldman, Dalibard, et al. 2013).

1 Classical cyclotron motion

1-1 Uniform circular motion

When a particle of mass M and charge q , described by Newtonian physics, is placed in a magnetic field, its motion is analyzed by solving the fundamental equation of dynamics $M\ddot{\mathbf{r}} = \mathbf{F}_L$, where $\mathbf{F}_L$ is the Lorentz force [see for example Jackson (1998), chapter 12]:

$$\mathbf{F}_L = q \mathbf{v} \times \mathbf{B}. \quad (\text{II.1})$$

Let us take the field $\mathbf{B}$ aligned with the z axis, $\mathbf{B} = B \mathbf{u}_z$. The motion along the z axis is then uniform translational motion, $\ddot{z} = 0$, and we can restrict our analysis to the motion projected onto the xy plane. In this plane, the equation of motion is written:

$$M\dot{\mathbf{v}} = q \mathbf{v} \times \mathbf{B} : \quad \ddot{x} = \omega_c \dot{y}, \quad \ddot{y} = -\omega_c \dot{x}, \quad (\text{II.2})$$

where we have introduced the *cyclotron angular frequency*

$$\omega_c = \frac{qB}{M}. \quad (\text{II.3})$$

The system of two differential equations (II.2) is easily solved to give

$$x(t) = X_0 - \frac{v_0}{\omega_c} \cos(\omega_c t), \quad y(t) = Y_0 + \frac{v_0}{\omega_c} \sin(\omega_c t), \quad (\text{II.4})$$

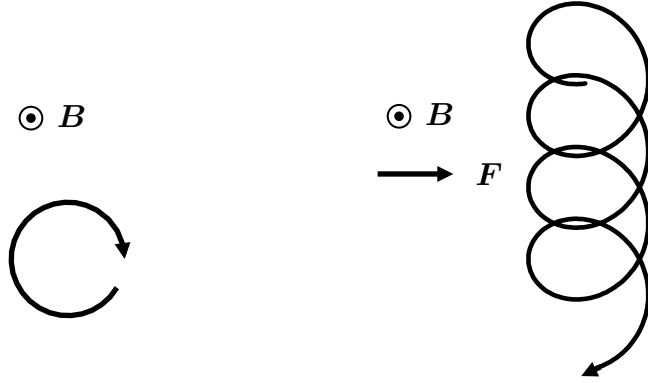


Figure II.1. Left: cyclotron orbit in a uniform magnetic field. Right: drift motion along y in the presence of an external force $\mathbf{F}$ parallel to the x axis.

which corresponds to uniform circular motion with angular frequency ω_c , traversed clockwise if $\omega_c > 0$. In (II.4), the origin of time has been chosen at an instant when the velocity vector is parallel to the y axis. The center (X_0, Y_0) of this circle is arbitrary, and its radius is

$$r_0 = v_0 / \omega_c. \quad (\text{II.5})$$

This relation can also be recovered by equating the Lorentz force and the centrifugal force: $qv_0B = Mv_0^2/r_0$.

The translational invariance of the problem is manifest in the general result (II.4): for a given initial velocity, the parameters of the circular trajectory (radius and angular frequency) will be the same regardless of the initial position in the plane. One can thus consider that applying a magnetic field to the particle amounts to localizing it in the phase space associated with the motion in the xy plane, its position and velocity being confined to a region centered at (X_0, Y_0) with extent

$$[\Delta r \times \Delta p]^2 \sim [r_0 \times (Mv_0)]^2 \quad \text{with} \quad r_0 = v_0 / \omega_c. \quad (\text{II.6})$$

For a given initial velocity v_0 , this region is smaller the larger the magnetic field B . Quantum mechanics, with its constraint linked to Heisenberg's inequality $\Delta r_i \Delta p_i \geq \hbar/2$ ($i = x, y$), will modify this simple result.

1-2 The Hall current

Let us now suppose that a uniform force $\mathbf{F}$, independent of position and velocity, is added to the Lorentz force $\mathbf{F}_L$. This force could, for example, be created by a uniform electric field. To simplify notation, let us take this force parallel to the x axis: $\mathbf{F} = F \mathbf{u}_x$. This force does not change the uniform character of the motion along z , and the equation of motion projected onto the xy plane becomes

$$M\ddot{x} = qB\dot{y} + F, \quad M\ddot{y} = -qB\dot{x}. \quad (\text{II.7})$$

In the absence of a magnetic field, the effect of the force is of course to accelerate the particle along the x axis. In the presence of a magnetic field, the nature of the motion changes radically. The general solution of the equation of motion (II.7) is

$$x(t) = X_0 - \frac{v_0}{\omega_c} \cos(\omega_c t), \quad y(t) = Y_0 + \frac{v_0}{\omega_c} \sin(\omega_c t) - \frac{F}{qB} t. \quad (\text{II.8})$$

This solution, shown in figure II.1, corresponds to the combination

- of the uniform circular motion corresponding to the cyclotron motion found in the absence of the force $\mathbf{F}$,
- of the uniform translational motion (drift motion) at velocity $-F/(qB)$, in the direction y perpendicular to the force $\mathbf{F}$.

When the force F is due to an external electric field ($\mathbf{F} = q\mathbf{E}$), this drift motion is called the *Hall current*.

2 The energy spectrum in quantum physics

Before turning to the quantitative description of the motion of a charged particle in a magnetic field, one can use the previous classical results, combined with Heisenberg's inequality, to predict the position and velocity scales of the problem. Let us start from the relation between position and velocity given in (II.5), $\omega_c r_0 = v_0$; using the fact that in a ground state of quantum motion one generally has

$$M \Delta x \Delta v \sim \hbar, \quad (\text{II.9})$$

we deduce the relevant length and velocity scales for a cyclotron orbit in quantum physics:

$$\ell = \sqrt{\frac{\hbar}{M\omega_c}} = \sqrt{\frac{\hbar}{qB}}, \quad v_m = \sqrt{\frac{\hbar\omega_c}{M}}. \quad (\text{II.10})$$

The length ℓ is called the *magnetic length*. For a free electron¹ in a magnetic field of 1 Tesla, one finds $\omega_c/2\pi = 28$ GHz, $\ell \approx 26$ nm, and $v_m \approx 4000$ m/s.

There are many ways to proceed to the quantitative stage and find the energies and eigenstates of the Hamiltonian of a particle in a uniform field $\mathbf{B}$:

$$\hat{H} = \frac{(\hat{\mathbf{p}} - q\mathbf{A}(\hat{\mathbf{r}}))^2}{2M}, \quad \text{with } \nabla \times \mathbf{A} = B\mathbf{u}_z. \quad (\text{II.11})$$

One reason for this great variety of techniques lies in gauge invariance: vector potentials $\mathbf{A}$ that are *a priori* very different (the Landau gauge or the symmetric gauge, for example) can give rise to correspondingly different-looking calculations, even though they ultimately lead to the same energy spectrum and the same eigenspaces. Since we will need these techniques to describe more complex situations later in the course (in the presence of an external potential, for example), we will now review them.

2-1 The spectrum obtained by the algebraic method

This method makes it possible to find the spectrum of the Hamiltonian without making any particular choice of gauge. Let us first consider the Hermitian *kinetic momentum* operator:

$$\hat{\Pi}_j = \hat{p}_j - qA_j(\hat{\mathbf{r}}), \quad j = x, y, \quad \hat{\mathbf{p}} = -i\hbar\nabla, \quad (\text{II.12})$$

with which the Hamiltonian can be rewritten as

$$\hat{H} = \frac{1}{2M} (\hat{\Pi}_x^2 + \hat{\Pi}_y^2) \quad (\text{II.13})$$

Contrary to what happens for the two components x, y of the position operator $\hat{\mathbf{r}}$ or the momentum operator $\hat{\mathbf{p}}$, the two components of the operator

¹For an electron in the conduction band of GaAs, the mass M , and hence the cyclotron angular frequency ω_c , are considerably modified: $M_b \approx 0.07 M$. For a detailed discussion, including magnetic spin effects, see for example Eisenstein (2005).

$\hat{\Pi}$ do not commute:

$$[\hat{\Pi}_x, \hat{\Pi}_y] = i\hbar qB. \quad (\text{II.14})$$

Finding the eigenstates of $(\hat{\Pi}_x^2 + \hat{\Pi}_y^2)/2M$, given that the commutator of these two Hermitian operators is a constant, is a problem very similar to that of diagonalizing the Hamiltonian of a harmonic oscillator $(\hat{P}^2 + \hat{X}^2)/2$ with $[\hat{X}, \hat{P}] = i$. We will therefore use a method similar to the one developed by Dirac to solve the harmonic-oscillator problem.

Let us introduce the two annihilation and creation operators

$$\hat{a} = \frac{1}{\sqrt{2\hbar qB}} (\hat{\Pi}_x + i\hat{\Pi}_y), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar qB}} (\hat{\Pi}_x - i\hat{\Pi}_y). \quad (\text{II.15})$$

The commutation relation

$$[\hat{a}, \hat{a}^\dagger] = 1 \quad (\text{II.16})$$

and the expression for the Hamiltonian

$$\hat{H} = \hbar\omega_c \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \quad (\text{II.17})$$

are identical to those obtained for the operators of a harmonic oscillator of angular frequency ω_c . We deduce that the eigenvalues of $\hat{H}$ are necessarily of the form

$$E_n = \left(n + \frac{1}{2} \right) \hbar\omega_c, \quad \text{with } n \text{ a non-negative integer.} \quad (\text{II.18})$$

These energy levels, regularly spaced by the amount $\hbar\omega_c$, are called *Landau levels*.

To find the form of the eigenstates, one must choose a gauge to express the vector potential, which we will do in the following paragraphs. Let us note already that, since we are studying a two-dimensional problem here and the energies are found to be labeled by a single quantum number n , one can expect the energy levels to be degenerate. We will see that this is indeed the case, the degeneracy being in fact macroscopic for each level E_n : the number of eigenstates grows proportionally to the area accessible to the particle in the xy plane.

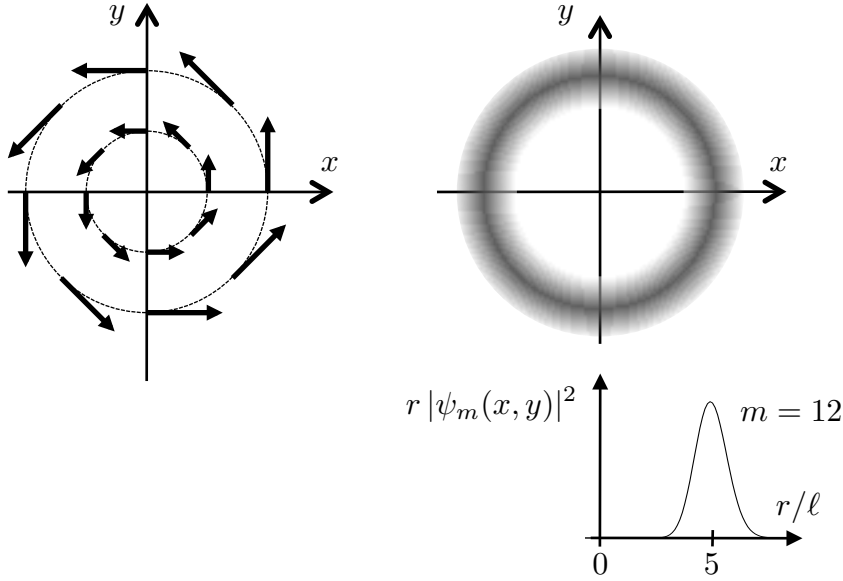


Figure II.2. Left: representation of the vector potential in the symmetric gauge. Right: probability density for an eigenfunction of the Hamiltonian in the symmetric gauge belonging to the LLL. The probability density $r|\psi_m(x, y)|^2$ is maximal for $r_m = \sqrt{2m+1}\ell$; this probability density is plotted here for $m = 12$.

2-2 Eigenstates and eigenenergies in the symmetric gauge

In this paragraph, we take the symmetric gauge (cf. figure II.2)

$$A_x(\mathbf{r}) = -By/2, \quad A_y(\mathbf{r}) = Bx/2, \quad A_z(\mathbf{r}) = 0. \quad (\text{II.19})$$

Note that this choice of gauge breaks the translational invariance of the problem, but respects the invariance under rotation about the center. The Hamiltonian $\hat{H}$ then expands to give:

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2M} + \frac{1}{8}M\omega_c^2(\hat{x}^2 + \hat{y}^2) - \frac{\omega_c}{2}(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x). \quad (\text{II.20})$$

which can again be written in the form

$$\hat{H} = \hat{H}_0 - \frac{\omega_c}{2}\hat{L}_z, \quad \text{with} \quad \hat{H}_0 = \frac{\hat{p}^2}{2M} + \frac{1}{8}M\omega_c^2 r^2, \quad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x. \quad (\text{II.21})$$

Let us first discuss separately the two components of $\hat{H}$, namely $\hat{H}_0$ and $\hat{L}_z$. The Hamiltonian $\hat{H}_0$ describes a two-dimensional isotropic harmonic oscillator, of angular frequency $\omega_c/2$. A basis of eigenstates of $\hat{H}_0$ can be sought by writing $\hat{H}_0 = \hat{H}_{0,x} + \hat{H}_{0,y}$, where $\hat{H}_{0,j}$ is the Hamiltonian of a one-dimensional oscillator along the direction $j = x, y$. A basis of eigenstates is then obtained by considering the states $|n_x, n_y\rangle$, in which n_j quanta of excitation have been deposited along the direction j , the associated energy being:

$$\left(n_x + \frac{1}{2}\right) \frac{\hbar\omega_c}{2} + \left(n_y + \frac{1}{2}\right) \frac{\hbar\omega_c}{2} = (n_0 + 1) \frac{\hbar\omega_c}{2}, \quad n_0 = n_x + n_y. \quad (\text{II.22})$$

In particular:

- the ground state $|0, 0\rangle$, of energy $\hbar\omega_c/2$, is non-degenerate;
- the first excited level, of energy $3\hbar\omega_c/2$, has degeneracy 2 and is spanned by the two states $|1, 0\rangle$ and $|0, 1\rangle$;
- the second excited level, of energy $5\hbar\omega_c/2$, has degeneracy 3 and is spanned by the three states $|2, 0\rangle$, $|1, 1\rangle$, and $|0, 2\rangle$, etc.

More generally, the degeneracy of level n_0 is $n_0 + 1$ (see figure II.3, top panel). The wave function associated with a state $|n_x, n_y\rangle$ is $P_{n_x}(x)e^{-x^2/4\ell^2} P_{n_y}(y)e^{-y^2/4\ell^2}$, where the P_n are the Hermite polynomials (of degree n).

The operator $\hat{L}_z$ is the z component of the orbital angular momentum. Its expression in polar coordinates is simple

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \varphi} \quad (\text{II.23})$$

and its eigenfunctions are thus of the type $F(r) e^{im\varphi}$, where m is a (signed) integer and F is an arbitrary function of the radial variable. Note that the

ground state of $\hat{H}_0$, with wave function $e^{-r^2/4\ell^2}$, is indeed of this form with $m = 0$.

Let us now look for a joint eigenbasis of $\hat{H}_0$ and $\hat{L}_z$, which will therefore also constitute an eigenbasis of $\hat{H}$. We can work within each eigenspace of $\hat{H}_0$ in turn:

- We have just noted that the ground state $e^{-r^2/4\ell^2}$ is a joint eigenstate of $\hat{H}_0$ and $\hat{L}_z$.
- In the first excited level of $\hat{H}_0$, the states $|n_x = 1, n_y = 0\rangle$ and $|n_x = 0, n_y = 1\rangle$ have, respectively, the wave functions $x e^{-r^2/4\ell^2}$ and $y e^{-r^2/4\ell^2}$; they are thus not individually eigenstates of $\hat{L}_z$, but it is easy to find two suitable linear combinations:

$$(x - iy) e^{-r^2/4\ell^2} = r e^{-i\varphi} e^{-r^2/4\ell^2} \quad \text{and} \quad (x + iy) e^{-r^2/4\ell^2} = r e^{+i\varphi} e^{-r^2/4\ell^2}, \quad (\text{II.24})$$

associated respectively with $m = -1$ and $m = +1$.

- More generally, within a given eigenspace of $\hat{H}_0$ associated with the quantum number $n_0 = n_x + n_y$, one can identify $n_0 + 1$ independent eigenstates of $\hat{L}_z$ associated with the azimuthal quantum numbers

$$m = -n_0, -n_0 + 2, \dots, n_0 - 2, n_0. \quad (\text{II.25})$$

The general expression for these states is fairly complicated, except for the two extreme states $m = \pm n_0$, which have wave functions:

$$m = \pm n_0 : \quad (x \pm iy)^m e^{-r^2/4\ell^2} = r^m e^{\pm im\varphi} e^{-r^2/4\ell^2} \quad (\text{II.26})$$

Now that we have a joint basis for $\hat{H}_0$ and $\hat{L}_z$, the problem of diagonalizing $\hat{H}$ is solved. A joint eigenstate of $\hat{H}_0$ and $\hat{L}_z$, with eigenvalues $(n_0 + 1)\hbar\omega_c/2$ and $m\hbar$, will also be an eigenstate of $\hat{H}$ with eigenvalue

$$E_{n_0, m} = (n_0 + 1) \frac{\hbar\omega_c}{2} - m \frac{\hbar\omega_c}{2}, \quad \text{with} \quad m = -n_0, -n_0 + 2, \dots, n_0 - 2, n_0. \quad (\text{II.27})$$

which can also be written

$$E_n = \left(n + \frac{1}{2}\right) \hbar\omega_c, \quad \text{with} \quad n = \frac{n_0 - m}{2} \text{ a non-negative integer.} \quad (\text{II.28})$$

We thus recover the structure of the Hamiltonian's regularly spaced Landau levels (see also figure II.3, bottom panel). We further note that each Landau level is *infinitely degenerate*, at least if the particle can explore the whole xy plane. Indeed, there is an infinite number of ways of reaching any given integer n from pairs (n_0, m) , if the achievable values of n_0 and m are not restricted.

2-3 The lowest Landau level (LLL)

This procedure provides a systematic way of finding a basis of eigenstates of $\hat{H}$. Let us now focus on the lowest Landau level $n = 0$ (*Lowest Landau Level, LLL*), obtained by systematically taking $n_0 = m$ in (II.28). We gave in (II.26) the form of the corresponding wave functions, which we reproduce here:

$$\psi_{n_0=m, m}(x, y) \propto (x + iy)^m e^{-r^2/4\ell^2} = r^m e^{im\varphi} e^{-r^2/4\ell^2}. \quad (\text{II.29})$$

We have plotted in figure II.2 the radial probability density $r|\psi_m|^2$ associated with a state ψ_m of the LLL (let us drop the index n_0 , since it is in any case equal to m for the LLL). This probability density is invariant under rotation about the z axis, and it is maximal on a circle of radius

$$r_m = \sqrt{2m+1} \ell. \quad (\text{II.30})$$

Its full width at half maximum is independent of m and of order

$$\Delta r \sim \ell. \quad (\text{II.31})$$

A particle prepared in a given state ψ_m is thus localized on a narrow ring, the union of all these rings covering the plane. One also finds

$$\langle \psi_m | r^2 | \psi_m \rangle = (2m+2)\ell^2. \quad (\text{II.32})$$

The orthogonality² of the different ψ_m is ensured by their different dependence on the polar angle φ :

$$\int \psi_m^*(\mathbf{r}) \psi_{m'}(\mathbf{r}) d^2r \propto \int e^{-im\varphi} e^{im'\varphi} d\varphi = 2\pi \delta_{m, m'}. \quad (\text{II.33})$$

²The normalized ψ_m are given by $\psi_m(\mathbf{r}) = N_m r^m e^{im\varphi} e^{-r^2/4\ell^2}$ with $N_m = [n! \pi (2\ell)^{n+1}]^{1/2}$.

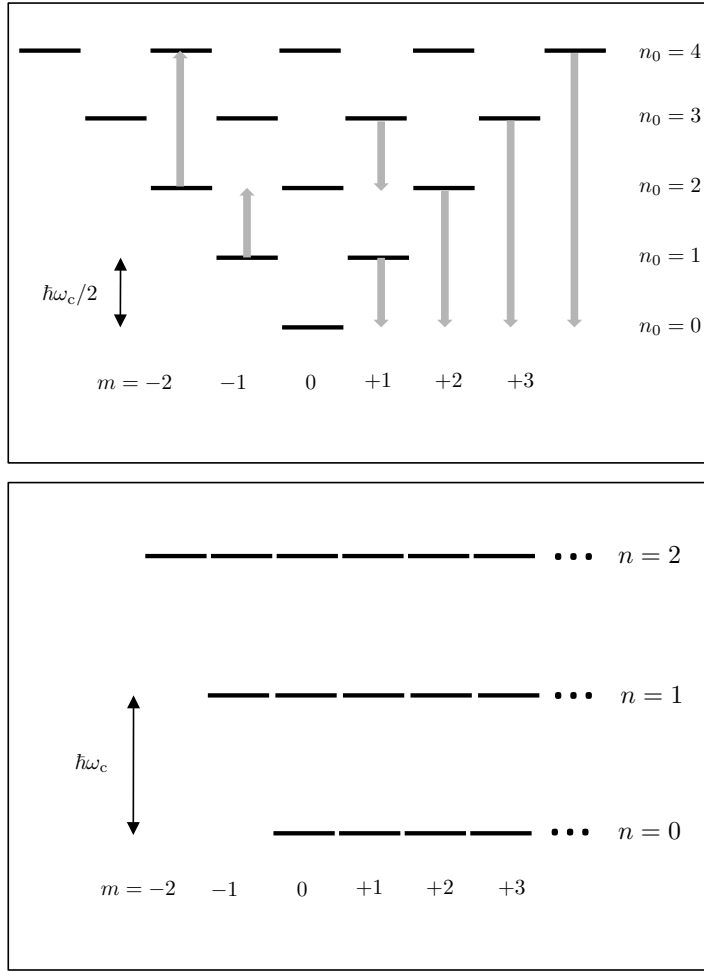


Figure II.3. Top: energy-level structure of the Hamiltonian $\hat{H}_0$ describing a 2D harmonic oscillator, with levels $(n_0 + 1)\hbar\omega_c/2$, n_0 a non-negative integer. The degeneracy of a level is $n_0 + 1$, and the eigenstates can be labeled by the two quantum numbers n_0 and m , where $m\hbar$ is the value of the z component of the angular momentum $\hat{L}_z$. The grey arrows indicate the action of $-\omega_c \hat{L}_z/2$ on certain levels. Bottom: energy-level structure of the Hamiltonian $\hat{H}$ describing a particle in a uniform magnetic field. This structure is deduced from the one above by shifting a level (n_0, m) by the energy $-\hbar\omega_c m/2$.

Let us now introduce the variable

$$u = x + iy. \quad (\text{II.34})$$

We deduce from (II.29) that an arbitrary state of the lowest Landau level can be written in the form

$$\psi(x, y) = \sum_m C_m \psi_m(x, y) = F(u) e^{-r^2/4\ell^2} \quad (\text{II.35})$$

where

$$F(u) = \sum_m C_m u^m \quad (\text{II.36})$$

is an arbitrary polynomial (or analytic function) of the variable u . Restriction to the lowest Landau level thus corresponds to an effective passage from a two-dimensional space to a one-dimensional space. Before restriction to the LLL, the possible states are wave functions of the two independent variables x and y , or equivalently, functions of the two complex variables $u = x + iy$ and $\bar{u} = x - iy$. After restriction to the LLL, the states are described by functions of the single variable $u = x + iy$ and no longer depend on $\bar{u}$ (except for the Gaussian factor $e^{-r^2/4\ell^2} = e^{-u\bar{u}/4\ell^2}$).

What orbital angular momentum for an LLL state? The fact that the basis states (II.29) are eigenstates of the canonical angular momentum $\hat{L}_z$, with all different eigenvalues $m\hbar$, might suggest that they do not all have the same average “physical” angular momentum. This is not the case; just as the momentum $\hat{\mathbf{p}}$ is not equal to the *kinetic momentum* $M\hat{\mathbf{v}}$, the operator $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$ does not describe the physical quantity *orbital angular momentum*³ $M\hat{\mathbf{r}} \times \hat{\mathbf{v}} = \hat{\mathbf{r}} \times \hat{\mathbf{\Pi}}$.

Let us compute the average value of the z component of this physical quantity. Starting from

$$\mathbf{r} \times \mathbf{\Pi} = \mathbf{r} \times \mathbf{p} - q\mathbf{r} \times \mathbf{A} = \mathbf{L} - \frac{q}{2}(x^2 + y^2)\mathbf{B} \quad (\text{II.37})$$

and using (II.32), we find

$$\langle \psi_m | (\mathbf{r} \times \mathbf{\Pi})_z | \psi_m \rangle = m\hbar - (m + 1)\hbar = -\hbar, \quad (\text{II.38})$$

³In English, $\mathbf{L}$ is called the *canonical angular momentum* and $\mathbf{r} \times \mathbf{\Pi}$ the *kinetic angular momentum*.

which gives an average orbital angular momentum equal to $-\hbar$ for all the states of the LLL.

Back to the algebraic method. The form $\hat{H} = (\hat{a}^\dagger \hat{a} + 1/2)\hbar\omega_c$ of the Hamiltonian obtained by the algebraic method (cf. (II.17)) makes it possible to simply recover the structure of the LLL states. For the choice of the symmetric gauge, the lowest level corresponds to the states $|\psi\rangle$ such that

$$\hat{a}|\psi\rangle = 0, \quad \text{with} \quad \hat{a} = \frac{1}{\sqrt{2\hbar qB}} \left(\hat{p}_x + i\hat{p}_y + \frac{qB}{2}(\hat{y} - i\hat{x}) \right). \quad (\text{II.39})$$

Let us re-express this operator in terms of the variables u and $\bar{u}$, treated as independent, and their derivatives:

$$u = x + iy, \quad \bar{u} = x - iy \quad \Rightarrow \quad \partial_u = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\bar{u}} = \frac{1}{2}(\partial_x + i\partial_y). \quad (\text{II.40})$$

We have

$$\hat{a} = -i\sqrt{\frac{2\hbar}{qB}} \left(\partial_{\bar{u}} + \frac{u}{4\ell^2} \right) \quad (\text{II.41})$$

and the functions $\psi(x, y) \equiv \psi(u, \bar{u})$ solving (II.39) are such that

$$\hat{a}\psi(u, \bar{u}) = 0 \quad \Rightarrow \quad \psi(u, \bar{u}) = F(u) e^{-u\bar{u}/4\ell^2} \quad (\text{II.42})$$

which indeed gives back (II.35).

Let us finally note that it is then possible to construct the states of the first excited Landau level (and then the other Landau levels) by letting the operator $a^\dagger$ act on the LLL states we have just found.

2-4 Degeneracy of the LLL.

Suppose that the motion of the particle in the xy plane is restricted to a domain of area S . We will of course assume that $S \gg \ell^2$, so that several cyclotron orbits can fit within it. The question related to the degeneracy is thus: how many independent orbits can be fit into it, or equivalently, what is the degeneracy of the LLL as a function of S ? The rigorous answer to this question is not simple. One must go from the plane to a sphere

or a torus and take the appropriate boundary conditions [see for example Cooper (2008)].

We will content ourselves here with an approximate argument, taking a disk-shaped domain of radius R , with $S = \pi R^2$. As before, we use the symmetric gauge, taking the center of the disk as the origin $\mathbf{r} = 0$. The radial probability density $r|\psi(x, y)|^2$ for a basis function (II.29) is maximal at radius $r = \sqrt{2m+1}\ell$ and decreases rapidly (over a distance $\sim \ell$) on either side of this radius. For this basis function to contribute significantly to the expansion of a physical state of the particle, the region where it takes a significant value must lie inside the disk, which imposes:

$$n < \frac{R^2}{2\ell^2}. \quad (\text{II.43})$$

The degeneracy $\mathcal{D}$ of the LLL is thus given by this number

$$\mathcal{D} \approx \frac{R^2}{2\ell^2} = \frac{S}{2\pi\ell^2} = \frac{S}{2\pi\hbar/qB} = \frac{\Phi}{\Phi_0}, \quad (\text{II.44})$$

where $\Phi = BS$ is the flux of the field through the area S and Φ_0 is the *flux quantum*:

$$\Phi_0 = \frac{h}{q}. \quad (\text{II.45})$$

In other words, one can consider that the area occupied by a cyclotron orbit is $2\pi\ell^2 = h/qB$, and that $\mathcal{D}$ represents the number of independent orbits that can be fitted into the area S .

We gave above the value $\ell = 26 \text{ nm}$ for an electron in a magnetic field of 1 T. For a typical sample of $100 \mu\text{m}^2$, this corresponds to a degeneracy $\mathcal{D} = 24\,000$.

3 Landau gauge and probability current

3-1 Separation of the Hamiltonian in the Landau gauge

For the rest of this course, it will be useful to consider the eigenvalue problem for the Hamiltonian in another commonly used gauge, the *Landau*

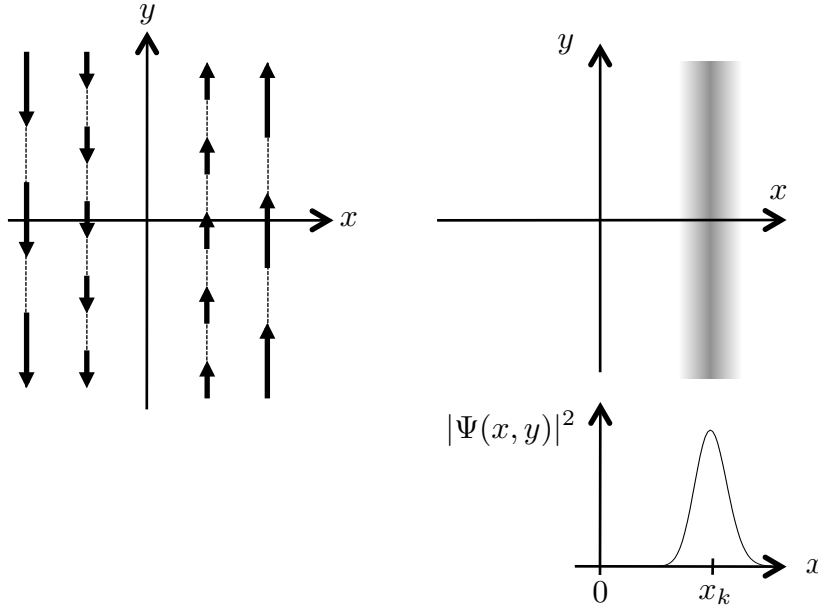


Figure II.4. Left: representation of the vector potential in the Landau gauge. Right: probability density for an eigenfunction of the Hamiltonian in the Landau gauge belonging to the LLL, $\Psi(\mathbf{r}) = \psi_k(x) e^{iky}$, where $\psi(x)$ is a Gaussian of width ℓ centered at $x_k = \hbar k / (qB) = k\ell^2$.

gauge, defined by (see figure II.4)

$$\mathbf{A} = Bx \mathbf{u}_y. \quad (\text{II.46})$$

The Hamiltonian of the particle placed in the magnetic field $\mathbf{B}$ is then written

$$\hat{H} = \frac{\hat{p}_x^2}{2M} + \frac{(\hat{p}_y - qB\hat{x})^2}{2M}. \quad (\text{II.47})$$

Motion along the y axis. Since the Hamiltonian (II.47) does not depend on the position operator $\hat{y}$, but only on the momentum operator $\hat{p}_y$, we

can look for its eigenstates in the form of plane waves e^{iky} , multiplied by a function ψ_k , at this stage unknown, of the variable x :

$$\hat{H}\Psi_k = E\Psi_k \quad \text{with} \quad \Psi_k(\mathbf{r}) = \psi_k(x) e^{iky}. \quad (\text{II.48})$$

It is convenient to assume that the sample has a finite size L_y along the y axis and to take periodic boundary conditions along this axis⁴. In this case, the wave number k describing the motion along y is quantized

$$k = \frac{2\pi}{L_y} n_y, \quad \text{with} \quad n_y \in \mathbb{Z}. \quad (\text{II.49})$$

Motion along the x axis. The function $\psi_k(x)$ introduced in (II.48), describing the motion along the x axis, is a solution of

$$-\frac{\hbar^2}{2M} \psi_k''(x) + \frac{(\hbar k - qBx)^2}{2M} \psi_k(x) = E \psi_k(x) \quad (\text{II.50})$$

which can be put in a more suggestive form:

$$-\frac{\hbar^2}{2M} \psi_k''(x) + \frac{1}{2} M \omega_c^2 (x - x_k)^2 \psi_k(x) = E \psi_k(x) \quad (\text{II.51})$$

with

$$x_k = \frac{\hbar k}{qB} = k\ell^2. \quad (\text{II.52})$$

We recognize the eigenvalue equation of a harmonic oscillator, with angular frequency ω_c , centered at the point x_k .

3-2 The Landau spectrum recovered

Since we have reduced the Hamiltonian to that of a one-dimensional harmonic oscillator, we of course know its spectrum:

$$E_{k,n} = \left(n + \frac{1}{2}\right) \hbar \omega_c. \quad (\text{II.53})$$

We indeed recover the result obtained by the algebraic method (cf. (II.18)). Several remarks can be made on this result:

⁴This formally amounts to considering a cylindrical surface pierced by a radial magnetic field, rather than a flat surface.

- Although the wave number k associated with the motion along y is generally nonzero, there is no kinetic energy $\hbar^2 k^2 / 2M$ associated with this wave number! The presence of the magnetic field means that the “only” effect of this wave number is to shift the center of the oscillator along the x direction. We will come back to this point further on, by evaluating the average velocity along y for a particle in the state Ψ_k .
- The LLL is obtained by taking the wave functions of the different oscillators associated with the wave numbers k in their ground state; each of these wave functions is a Gaussian centered at x_k and of width ℓ . The basis of LLL functions obtained in this way is thus of the form

$$\Psi_k(\mathbf{r}) \propto e^{-(x-x_k)^2/2\ell^2} e^{iky}. \quad (\text{II.54})$$

- The orthogonality of the different functions Ψ_k thus found comes from their different dependence on the variable y :

$$\int \Psi_k^*(\mathbf{r}) \Psi_{k'}(\mathbf{r}) d^2r \propto \int e^{-iky} e^{ik'y} dy = L_y \delta_{n_y, n'_y}, \quad (\text{II.55})$$

where n_y and n'_y are the two integers characterizing the quantization of the wave numbers k and k' , according to the rule given in (II.49).

- Two consecutive values of k , separated by $2\pi/L_y$, lead to very close oscillator centers:

$$x_k - x_{k'} = 2\pi \frac{\ell^2}{L_y} \quad \text{if} \quad k - k' = \frac{2\pi}{L_y}, \quad (\text{II.56})$$

which is very small compared to ℓ if the sample size L_y is itself large compared to ℓ .

Degeneracy of the eigenvalues. We can finally recover the degeneracy of a given Landau level, for example the LLL. Let us place the particle in a rectangular box of size $L_x \times L_y$. For the function $\Psi(x, y)$ to be relevant, the center x_k of the oscillator must lie inside this box, which imposes a constraint between the wave number k and the size L_x

$$0 \leq x_k \leq L_x \quad \Rightarrow \quad 0 \leq k \leq \frac{L_x}{\ell^2}. \quad (\text{II.57})$$

Moreover, we have seen that the finite value of the size L_y requires that k can only take discrete values, spaced by $2\pi/L_y$ [cf. (II.49)]. This gives a total number of possible values, taking (II.57) into account:

$$\mathcal{D} \approx \frac{L_x}{\ell^2} \frac{L_y}{2\pi} = \frac{S}{2\pi\ell^2} \quad (\text{II.58})$$

where $S = L_x L_y$ represents the area of the rectangle. To each of these values of k is associated a relevant harmonic oscillator, whose ground state belongs to the LLL. We thus recover the degeneracy obtained for the symmetric gauge in (II.44).

3-3 Probability current in a Landau state

To understand why no kinetic-energy term $\hbar^2 k^2 / (2M)$ appears in the total energy of a state Ψ_k , we will evaluate the average kinetic momentum of a particle prepared in this state. Recall that the operator describing the physical quantity “kinetic momentum,” i.e., $M\mathbf{v}$, is written

$$\hat{\Pi} = \hat{\mathbf{p}} - q\mathbf{A}(\hat{\mathbf{r}}) \quad \text{with} \quad \hat{\mathbf{p}} = -i\nabla. \quad (\text{II.59})$$

Consider a given LLL state of the form given in (II.54):

$$\Psi_k(\mathbf{r}) = \psi_k(x) e^{iky} \propto e^{-(x-x_k)^2/2\ell^2} e^{iky} \quad (\text{II.60})$$

and let us examine the average kinetic momentum

$$M\langle \hat{\mathbf{v}} \rangle = \int \Psi_k^*(\mathbf{r}) (-i\hbar\nabla - q\mathbf{A}(\mathbf{r})) \Psi_k(\mathbf{r}) d^2r. \quad (\text{II.61})$$

The x component of this kinetic momentum is zero. Indeed, $A_x = 0$, and the average value of the momentum p_x in the ground state of the reduced Hamiltonian along x , given in (II.50), is also zero.

The reasoning along the y axis is more subtle. The action of $\hat{p}_y$ on $\Psi_k(\mathbf{r})$ brings out $\hbar k$, so that

$$Mv_y(k) = \int \Psi_k^*(\mathbf{r}) (\hbar k - qA_y) \Psi_k(\mathbf{r}) d^2r. \quad (\text{II.62})$$

This quantity can be computed explicitly, giving, after integration over y ,

$$v_y(k) = -\omega_c \int |\psi_k(x)|^2 (x - x_k) dx. \quad (\text{II.63})$$

For the ground state $\psi_k(x)$ of the harmonic oscillator centered at x_k , this integral is zero, since the ground state is symmetric about the point x_k . We thus find that, despite the presence of the e^{iky} term in the expression of the eigenstate, the average velocity – or the average kinetic momentum – associated with this state is zero: the contribution of the momentum operator is exactly cancelled by that of the vector potential.

3-4 The originality of the Landau-level structure

At this stage, we have thus fully characterized the spectrum of a charged quantum particle moving in an xy plane perpendicular to a magnetic field. We have found an energy spectrum made up of discrete, equally spaced levels, with a separation $\hbar\omega_c$. Each level is macroscopically degenerate. More precisely, the number of independent states making up each level, i.e., the dimension of the eigenspace, grows linearly with the area of the sample.

At first sight, this result is very similar to the one obtained for a particle placed in a periodic lattice, in the tight-binding limit (figure II.5). Let us liken each lattice site to a harmonic well of angular frequency ω ; in the absence of tunneling, the energy spectrum of the particle is made up of discrete states separated by the energy $\hbar\omega$. Putting together all these spectra for the different lattice sites gives a result formally identical to that of the Landau-level spectrum, with a degeneracy of each energy that grows linearly with the area (or the number of sites) of the system. In the presence of tunneling, the energy bands of the lattice acquire a certain width, but this width remains small as long as the tunneling matrix element is small compared to $\hbar\omega$.

We thus arrive at the following important question:

Is there a difference between the problem of a particle moving in a plane orthogonal to $\mathbf{B}$ and that of a particle evolving in a deep 2D lattice?

The answer is yes: the properties of a system are not reduced to the spectrum of its Hamiltonian, and the structure of the eigenspaces plays an

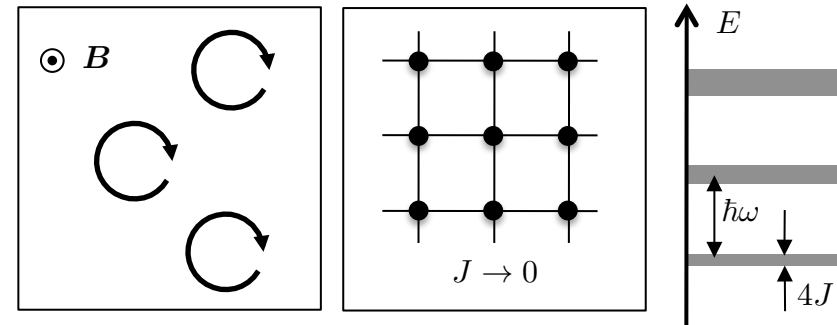


Figure II.5. Two physical systems leading to the same spectrum. Left, a particle in a uniform magnetic field. Right, a particle in a periodic lattice in the limit where the tunneling coupling J between neighboring sites tends to 0. The spectrum then consists of narrow bands separated by $\hbar\omega$, where ω is the angular frequency of the oscillation at each lattice site.

essential role. Even if one restricts oneself to the case of a single particle, Landau levels are not ordinary bands. They have a nontrivial topological property, characterized by a nonzero Chern number. The simplest manifestation of this property is obtained for a system of finite size, in which edge states of definite chirality are necessarily present. We will illustrate this point in the following paragraph.

4 Edge states and applications

Our goal in this paragraph is to show an essential feature of quantum orbital magnetism, namely the presence of edge states capable of carrying a current in a sample. A very important consequence of the existence of these states is the presence of plateaus in the integer quantum Hall effect, which we will very briefly mention. We will follow here an approach inspired by Girvin (2000), Yoshioka (2002), and Goerbig (2009).

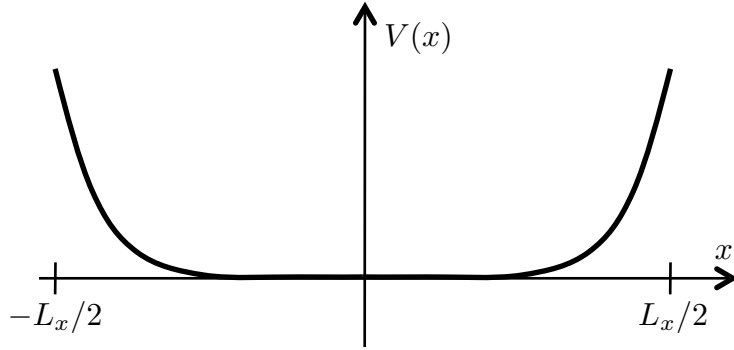


Figure II.6. Potential $V(x)$ confining the particles along the x direction.

4-1 Modeling a finite-size sample

We will consider a finite-size, rectangular sample. We will use the Landau gauge introduced in the previous paragraph:

$$\mathbf{A} = Bx \mathbf{u}_y, \quad (\text{II.64})$$

which gives different roles to the x and y directions. Along the y axis, the accessible domain has length y , and we will take, as before, periodic boundary conditions along this axis. Along the x axis, we assume that the particles are subject to a potential $V(x)$, confining the particle to the region between $-L_x/2$ and $L_x/2$. The exact shape of the potential does not matter much, and the only condition we will impose is that the distance over which $V(x)$ rises is small compared to L_x (see figure II.6).

The treatment we developed previously to find the spectrum of the Hamiltonian and a basis of eigenstates remains valid. One can still look for the eigenstates in the form

$$\Psi_k(\mathbf{r}) = \psi_k(x) e^{iky}, \quad k = \frac{2\pi}{L_y} n_y, \quad n_y \in \mathbb{Z}, \quad (\text{II.65})$$

where the wave function $\psi_k(x)$ is now a solution of

$$-\frac{\hbar^2}{2M} \psi_k''(x) + \left[\frac{1}{2} M \omega_c^2 (x - x_k)^2 + V(x) \right] \psi_k(x) = E \psi_k(x) \quad (\text{II.66})$$

with $x_k = k\ell^2$.

4-2 Edge states

To simplify our mathematical treatment, we will assume⁵ here that the potential $V(x)$ varies slowly on the length scale ℓ characterizing the extent of the ground state of the harmonic potential $M\omega_c^2 x^2/2$. At fixed k , we can then linearize the potential $V(x)$ near x_k :

$$V(x) \approx V(x_k) + (x - x_k)V'(x_k) \quad (\text{II.67})$$

which allows us to rewrite the eigenvalue equation (II.66) in the form

$$-\frac{\hbar^2}{2M} \psi_k''(x) + \left[\frac{1}{2} M \omega_c^2 (x - x_k - \delta x_k)^2 + \epsilon_k \right] \psi_k(x) = E \psi_k(x) \quad (\text{II.68})$$

where we have introduced the shift of the center of the harmonic oscillator induced by the force derived from the potential $V(x)$

$$\delta x_k = -\frac{V'(x_k)}{M\omega_c^2} \quad (\text{II.69})$$

and the energy shift

$$\epsilon_k = V(x_k) - \frac{1}{2} M \omega_c^2 (\delta x_k)^2 \approx V(x_k). \quad (\text{II.70})$$

This last approximation is valid if we restrict ourselves to potentials with edges soft enough that $\delta x_k \ll \ell$.

Let us now examine the average velocity associated with the eigenstates ψ_k , given the presence of the potential $V(x)$. If the potential $V(x)$ is chosen to have a flat bottom, the force $-V'(x)$ is zero over the whole extent of this flat region, and so is the shift δx_k .

- For states whose center x_k lies in this flat part of the potential, the argument developed in the previous paragraph remains valid: the average velocity, given by

$$v_y(k) = -\omega_c \int |\psi_k(x)|^2 (x - x_k) dx. \quad (\text{II.71})$$

is zero, because the probability density $|\psi_k(x)|^2$ is symmetric about x_k .

⁵This assumption is not essential, as explained for example in Yoshioka (2002).

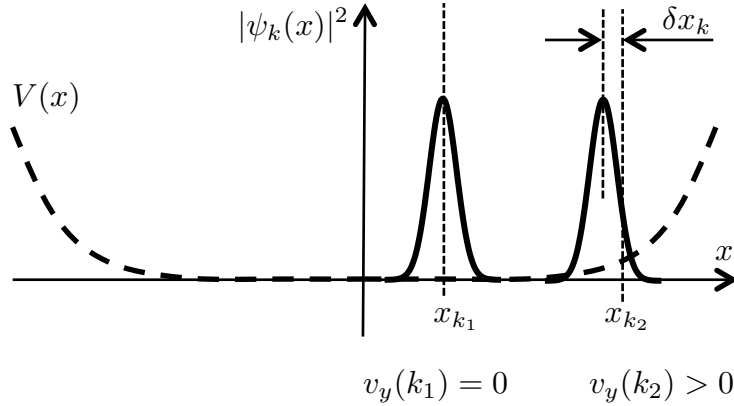


Figure II.7. Probability density of the eigenfunctions in the Landau gauge, in the presence of a potential $V(x)$. The centers of the wave functions are shifted by δx_k [cf. (II.69)] in the regions where the potential varies ($V'(x) \neq 0$). This shift of the center is associated with a nonzero average velocity along the y direction.

- On the other hand, for states whose center x_k is located at the edges of the sample,

$$x_k \approx \pm L_x/2, \quad (\text{II.72})$$

the force $-V'(x_k)$, and hence the shift δx_k , are nonzero. More precisely, the integral (II.71) gives, using the fact that ψ_k is normalized:

$$v_y(k) = -\omega_c (\langle x \rangle - x_k) = -\omega_c \delta x_k = \frac{V'(x_k)}{M\omega_c}. \quad (\text{II.73})$$

There is thus a nonzero velocity v_y associated with states located at the edges of the sample. For $\omega_c = qB/M > 0$, this velocity is negative for states near the edge $x = -L_x/2$ and positive on the other edge $x = L_x/2$. This therefore corresponds to a rotation of the particle along the edges of the sample in the direct (counterclockwise) sense, whereas the cyclotron motion occurs, under the same conditions, in the clockwise sense. The existence of these chiral edge states, whose velocity has a sign determined by the orientation of the magnetic field, constitutes an essential difference between the dynamics on a “usual” lattice and that obtained for Landau

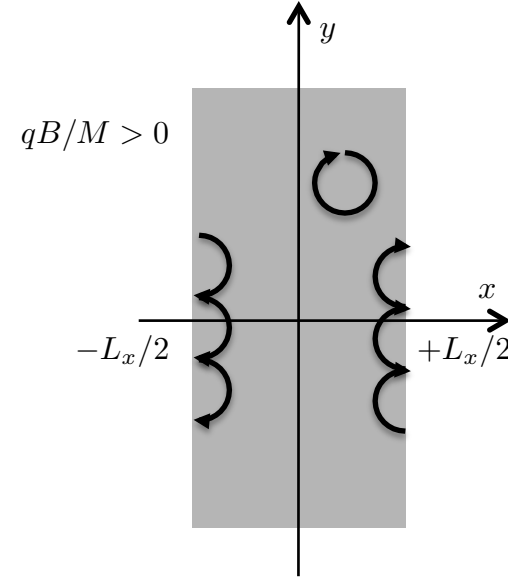


Figure II.8. Classical orbits corresponding to the quantum edge states.

levels. It is interpreted classically in terms of “skipping orbits.” These are cyclotron trajectories interrupted by a bounce off the edge of the sample (figure II.8).

One notes that the direction and magnitude of this edge-state velocity are directly related to the analysis we made earlier for the Hall current in the classical case. The confining potential creates a force $F = -V'$ along the x direction, and the particles acquire a drift velocity $-F/(M\omega_c) = -F/(qB)$ in the y direction [cf. (II.8)].

Let us indicate another way of recovering the velocity associated with an edge state. The energy of the state ψ_k solving the differential equation (II.68) is

$$E(k) = (n + 1/2)\hbar\omega_c + \epsilon_k \approx (n + 1/2)\hbar\omega_c + V(x_k). \quad (\text{II.74})$$

This relation can be viewed as a dispersion relation: the associated group

velocity is directed along y and equals

$$v_{1,y}(k) = \frac{1}{\hbar} \frac{dE}{dk} \approx \frac{1}{\hbar} \frac{dV(x_k)}{dk} = \frac{1}{\hbar} \left(\frac{dV}{dx} \right)_{x=x_k} \frac{dx_k}{dk} \quad (\text{II.75})$$

i.e.,

$$v_{1,y}(k) = \frac{1}{qB} \left(\frac{dV}{dx} \right)_{x=x_k} \quad (\text{II.76})$$

which is indeed identical to (II.73), since $\omega_c = qB/M$.

4-3 The edge current for fermions at $T = 0$

Suppose now that we have a gas of non-interacting, spinless fermions, and that this gas is at zero temperature (ground state). This ground state is obtained by filling all the single-particle levels up to the Fermi energy corresponding to the chemical potential μ . Let us focus on the case where the Fermi energy lies between two Landau levels, for example between the LLL, of energy $\hbar\omega_c/2$, and the first excited Landau level, of energy $3\hbar\omega_c/2$.

We will thus fill, on both edges of the sample, edge states whose energy varies between $\hbar\omega_c/2$ and μ . Let us try to evaluate the total current flowing along the edge of the sample. Each occupied state Ψ_k contributes to the particle current⁶ an amount $v_y(k)/L_y$, since the particle is delocalized all along the y axis. The current is obtained by summing the contribution of all the relevant edge states, for example for the right edge, around $x = L_x/2$:

$$J_{y,r} = \frac{1}{L_y} \sum_k v_y(k), \quad (\text{II.77})$$

where the sum runs over all the states k whose center x_k lies in the rising region of $V(x)$. The discrete sum over k can be replaced by an integral

$$\sum_k [\dots] \rightarrow \frac{L_y}{2\pi} \int [\dots] dk \quad (\text{II.78})$$

which leads to

$$J_{y,r} = \frac{1}{L_y} \frac{L_y}{2\pi} \frac{1}{M\omega_c} \int V'(x_k) dk = \frac{1}{\hbar} \int V'(x) dx \quad (\text{II.79})$$

⁶Here we speak of a particle current, not an electric current. The latter is obtained from J_y by multiplying it by the (negative) charge of the electron.

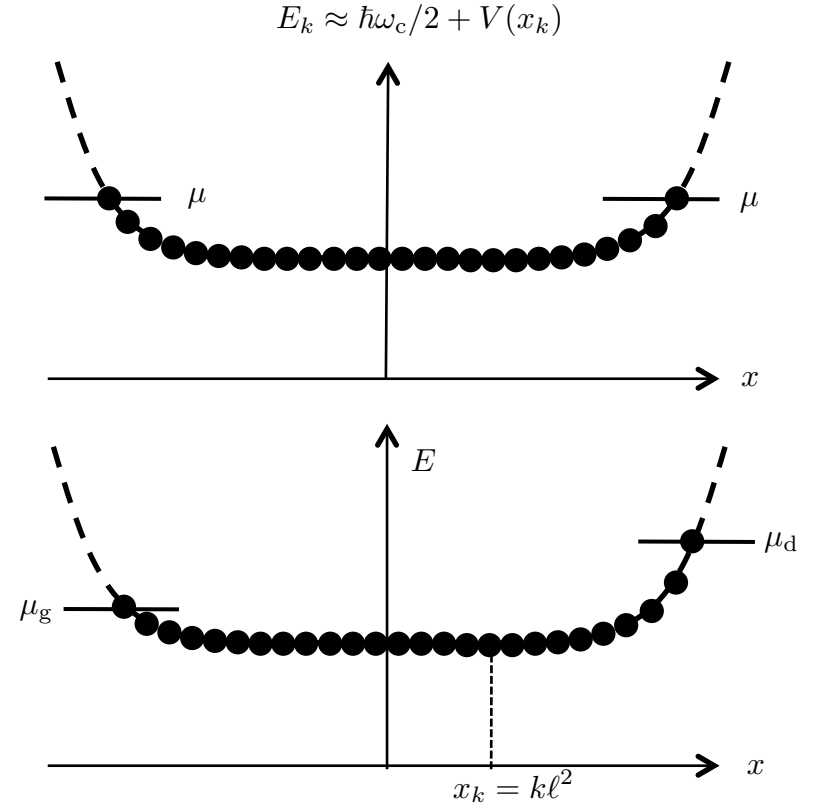


Figure II.9. Zero-temperature filling of the LLL states in the absence (top) and presence (bottom) of a chemical-potential difference between the left and right of the sample. Each point represents an occupied state Ψ_k , labeled here by its center x_k . We have assumed $\mu < 3\hbar\omega_c/2$, so that the other Landau levels can be neglected.

i.e., finally,

$$J_{y,r} = \Delta E/h, \quad (\text{II.80})$$

where $\Delta E = \mu - \hbar\omega_c/2$ represents the energy difference between the Fermi energy and the LLL energy in the flat part of the potential. One finds the same value of the current, with the opposite sign, for the other edge of the sample, at $x = -L_x/2$. The two currents thus cancel exactly, and this cancellation occurs even if the shape or steepness of the edges is not the same on the left and on the right.

This remarkable result indicates that the edge of the sample behaves like a calibrated conductor, carrying a current independent of its size and of the value of the magnetic field, provided B is large enough for the length ℓ to be microscopic and for the notion of edge states, on which the preceding treatment relies, to be valid⁷. We have restricted ourselves here to a rectangular sample, with particular boundary conditions, but this result can be generalized to a sample of arbitrary shape, the value (II.80) being universal.

4-4 Hall-effect circuitry

Let us keep the same geometry as before and suppose that the two-dimensional electron gas is inserted between two electrodes made of a conventional three-dimensional metal, located at the top and bottom of the rectangle. We inject a particle current J_y into the lower electrode (1) and recover this current in the upper electrode (2). The Fermi levels in the electrodes are thus different, with $\mu_1 > \mu_2$. Assuming perfect electrode–electron-gas contacts⁸, the edge states located on the right of the sample will fill up to the level $\mu_r = \mu_1$, and those on the left of the sample will fill up to $\mu_l = \mu_2$.

⁷Thanks to M.-O. Goerbig for an illuminating discussion on this subject!

⁸The underlying double assumption is (i) that electrons up to energy μ_1 are supplied to the right edge, as needed, by electrode 1, and (ii) that any electron of energy greater than μ_2 on the left edge is immediately transported to electrode 1, which absorbs it. The irreversibility associated with this double assumption manifests itself as energy dissipation in the two zones marked by a red circle: at the bottom left, where electrons of energy $\sim \mu_1$ enter edge states that contain only electrons of energy $\leq \mu_2$, and at the top right, where electrons with energy up to μ_1 enter the electrode at chemical potential μ_2 . For a study going beyond this “perfect electrodes” model, see for example Yoshioka (2002).

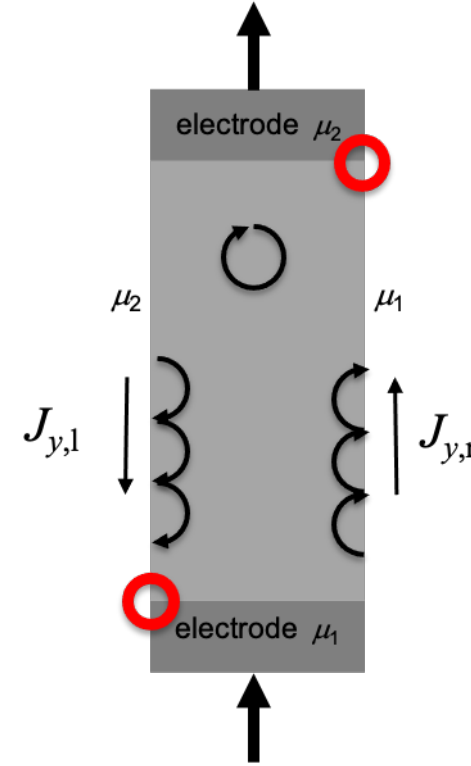


Figure II.10. Idealized diagram of a Hall-conductivity measurement experiment. Two “ideal” electrodes impose different chemical potentials on the left and right edges of the sample. This breaks the balance between the corresponding edge currents ($J_{y,r} \neq J_{y,l}$), and a current J_y flows in and out of the electrodes. The circled zones are those where a dissipative process occurs to keep the system in a steady-state regime.

To simplify the analysis, we continue to assume that only the LLL plays a role in what follows. The calculation of the previous paragraph continues to apply, but now with different edge currents on the left and right of the sample, which corresponds to the appearance of a nonzero net current:

$$\begin{aligned} J_y &= J_{y,r} - J_{y,l} \\ &= \frac{1}{h} (\mu_r - \hbar\omega_c/2) - \frac{1}{h} (\mu_l - \hbar\omega_c/2) \\ &= (\mu_r - \mu_l)/h. \end{aligned} \quad (\text{II.81})$$

If we go back to the electric current $I_y = qJ_y$ and the potential difference $V_x = (\mu_r - \mu_l)/q$, we obtain the relation

$$I_y = \sigma_{xy} V_x \quad \text{with} \quad \sigma_{xy} = q^2/h. \quad (\text{II.82})$$

corresponding to a *Hall resistance*

$$R_H = h/q^2 = 25\,812.807\,557(18) \, \Omega. \quad (\text{II.83})$$

This formula generalizes simply to the case where several Landau levels are filled and participate in conduction: one finds that the conductivity σ_{xy} is simply multiplied by the number of Landau levels involved. This value of the “conductance quantum,” equal to e^2/h , is identical to the result predicted for a one-dimensional conductor connecting two three-dimensional electrodes (Landauer–Büttiker formula, see for example Akkermans & Montambaux (2004) and Imry (1997)). One can thus regard the edge of a two-dimensional electron gas in the LLL as a one-dimensional conductor.

Remark. This result remains valid if the magnetic field is changed slightly around the value corresponding to figure II.9. Suppose, for example, that B is increased slightly. The distance between two adjacent centers x_k will decrease, and the number of available states $\mathcal{N}$ within the area $L_x \times L_y$ will thus increase

$$\mathcal{N} = \frac{qL_xL_y}{h} B \quad \longrightarrow \quad \mathcal{N} + \delta\mathcal{N} = \frac{qL_xL_y}{h} B + \frac{qL_xL_y}{h} \delta B \quad (\text{II.84})$$

If this change $\delta\mathcal{N}$ is smaller than the number of states located in the transition region where the potential $V(x)$ varies, figure II.9 remains valid, and

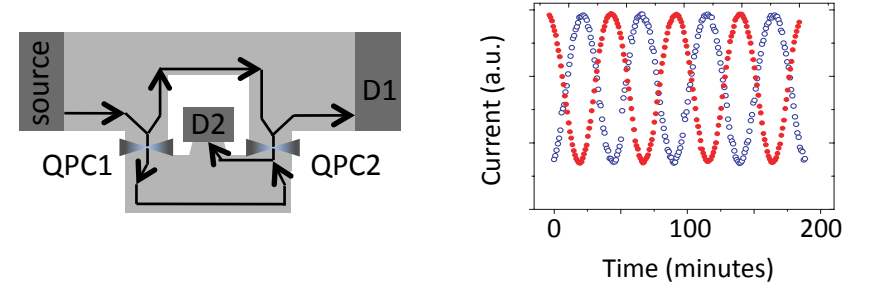


Figure II.11. Left: electronic Mach–Zehnder interferometer of Ji, Chung, et al. (2003). The sample is placed in a magnetic field of 5.5 T perpendicular to the plane of the figure. Right: in blue, signal observed as a function of time, related to the variation of the Aharonov–Bohm phase as the applied magnetic field decreases (relative variation over 3 hours: $7 \cdot 10^{-5}$). The signal in red is obtained by modifying the trajectories of the edge states using auxiliary electrodes. The area of the interferometer is about $60 \, \mu\text{m}^2$.

so does the result (II.83). One thus expects to find a plateau over which the Hall resistance keeps this value as B varies.

In practice, the plateau width deduced from this argument is much smaller than the plateaus actually observed. The reason is that the latter are considerably broadened by the disorder present in the samples. The presence of disorder in the potential felt by the particles within the bulk of the material makes it possible to produce a relatively large density of states localized in the gap between two Landau levels, for example $[\hbar\omega_c/2, 3\hbar\omega_c/2]$. This disorder does not destroy the quantization relation (II.82) as long as it remains weak enough not to create a connection between the two edges of the sample (see for example the detailed discussion by Girvin (2000)).

4-5 An electronic Mach–Zehnder interferometer

A beautiful application of this single-channel conductor provided by the edge states of a 2D electron gas under a magnetic field was demonstrated by Ji, Chung, et al. (2003). It is a two-path, Mach-Zehnder-type interfer-

ometer for electrons emitted by a source electrode (figure II.11). After traveling through the sample, these electrons are detected at one of the two electrodes $D1$ and $D2$. To go from the source to the detectors, two paths are possible for the electrons, with beam splitters formed by quantum point contacts (*Quantum Point Contact*, QPC), which can coherently transmit or reflect the electrons with an adjustable probability.

Because of the presence of the magnetic field perpendicular to the sample ($B = 5.5$ T), there is a phase difference between the two paths related to the Aharonov–Bohm effect seen in the previous lecture. This phase difference is proportional to the flux of the field through the circuit traced out by the edge states, and is of order $60\,000 \times 2\pi$.

Figure II.11 (right) shows an example of an interference signal; the current in detector $D1$ is plotted as a function of time. The modulation of the signal with time comes from the fact that the magnetic field applied to the sample decreases slowly (0.12 mT per hour). This is a very small change in relative terms over the duration of the experiment (about 7×10^{-5}), but one that changes the phase between the two arms by

$$7 \times 10^{-5} \times 60\,000 \times (2\pi) \approx 4 \times (2\pi), \quad (\text{II.85})$$

which corresponds to 4 complete oscillations of the signal. A detailed description of the possible decoherence mechanisms in this type of interferometer can be found in the article by Ji, Chung, et al. (2003).

4-6 Edge states with cold atoms?

We conclude this chapter with a brief discussion of the possibilities offered by cold-atom experiments for directly demonstrating these edge states. We will assume that it is possible to realize a Hamiltonian of the type $(\hat{p} - q\mathbf{A}(\hat{r}))^2/2M$ for neutral atoms, where $\mathbf{A}$ corresponds to a uniform magnetic field (this realization will be described in the following chapters). One can then demonstrate the edge states by abruptly changing the confinement of the gas and observing the subsequent trajectory of the atoms.

An example taken from the article by Goldman, Dalibard, et al. (2013) is shown in figure II.12. We start from fermionic atoms filling the LLL states in a certain region of space, namely a small disk located on the left

of the sample. At time $t = 0$, part of the confinement is abruptly released, and the atoms are then free to occupy a second disk, much larger than the first, these two disks being tangent. The Schrödinger equation is solved to determine the probability density at later times.

The results in figure II.12 confirm our intuition about edge states: the particles propagate chirally along the edge of the large disk, whereas direct migration toward the center of this large disk is almost nonexistent on the time scale considered. This absence of migration is a consequence of the fact that the LLL is flat, so that there is no quantum motion associated with a superposition involving only these states. The evolution shown in figure II.12 thus simultaneously reveals two essential features of the LLL: a flat spectrum for states localized in the bulk of the sample, and a chiral velocity for the edge states.

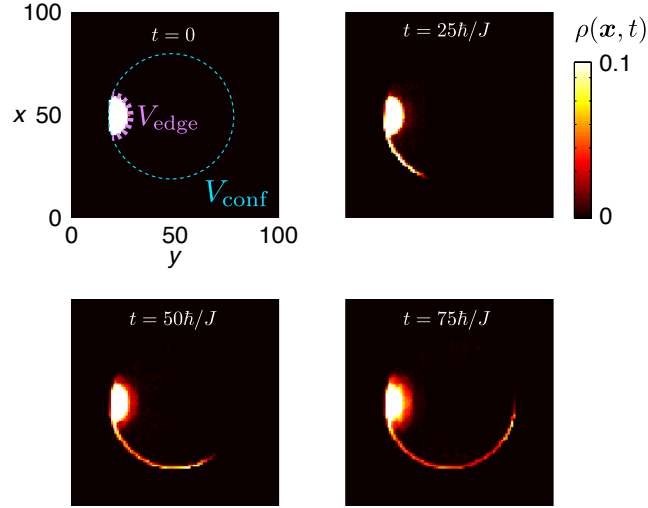


Figure II.12. Proposed demonstration of edge states in a cold-atom experiment. Fermionic atoms are initially confined in a small disk on the left. After part of the confinement is removed, the atoms can freely expand and occupy a much larger disk. These calculations are done for atoms moving on a square lattice of spacing a within the tight-binding approximation, with a flux of $2\pi/5$ per unit cell. The atoms are prepared in the lowest subband, which is a good approximation of the LLL (with a width nearly zero compared to the tunneling coefficient J). We took $\mu = -1.5 J$, and the radius of the large disk is $32 a$.

Chapter III

Berry phase and geometric gauge potentials

In physics, classical or quantum, it is common to make use of the notion of adiabatic evolution for a system whose state depends on external parameters – the volume of a box confining a fluid, external fields applied to the system, etc. Let us consider this type of evolution and focus in particular on a “closed trajectory” of the external parameters. By this we mean an evolution such that these external parameters, generically represented by a vector λ , evolve slowly from time 0 to time T , returning at time T to the value they had at time 0:

$$\lambda(0) \longrightarrow \lambda(t) \longrightarrow \lambda(T) = \lambda(0). \quad (\text{III.1})$$

It is then natural to think that the system under study is, at time T , in the same state as at time 0. However, the result is sometimes a little more subtle than this simple identity. A well-known example is the Foucault pendulum. Suppose that the pendulum oscillates linearly at the initial time at a point of given latitude. At time $T = 24$ hours, corresponding to one complete rotation of the Earth about its axis, the oscillation plane of the pendulum will not be the same. It will have rotated by an amount that depends on the latitude of the place where the experiment is performed.

The goal of this chapter is to take advantage of this “*anholonomy*”, i.e., this possibility for certain variables not to return to their initial value even though other variables, which drive the motion of the former, traverse a closed cycle. We will mainly be interested in the quantum case, which will allow us to bring out the notion of the Berry phase and thereby generate the equivalent of an Aharonov–Bohm phase, even if the particle carries no

electric charge.

1 The adiabatic approximation

1-1 Statement of the problem and notation

In this part we consider a quantum system described by a Hamiltonian $\hat{H}(\lambda)$ depending on a *continuous* parameter λ . For now we assume that this parameter is controlled by an operator external to the system. It may be a real number or a set of real numbers represented by a vector, hence the generic notation λ .

For each value of λ , we assume that the eigenstates $|\psi_n(\lambda)\rangle$ of $\hat{H}(\lambda)$ and their associated energies $E_n(\lambda)$ are known:

$$\hat{H}(\lambda)|\psi_n(\lambda)\rangle = E_n(\lambda)|\psi_n(\lambda)\rangle. \quad (\text{III.2})$$

We will assume in what follows that the $|\psi_n(\lambda)\rangle$ form an orthonormal basis of the Hilbert space for every value of λ . We will seek to determine the time evolution of the state vector $|\psi(t)\rangle$ of the system when the parameter λ itself depends on time. We will be particularly interested in slow evolutions of $\lambda(t)$ (in a sense that we will make precise).

Example 1. Consider a particle whose magnetic moment is described by the operator $\hat{\mu}$. We assume the particle to be at rest and localized at a point, so that its only degree of freedom is this magnetic moment. The magnetic field $\mathbf{B}$ at the location of the particle here plays the role of the parameter λ . The Hamiltonian is thus:

$$\lambda \equiv \mathbf{B}, \quad \hat{H}(\lambda) \equiv \hat{H}(\mathbf{B}) = -\hat{\mu} \cdot \mathbf{B}, \quad (\text{III.3})$$

the magnetic field $\mathbf{B}$ being allowed to depend on time, as in an NMR experiment.

Example 2. When describing the interaction of an atom with radiation, one often makes use of the *semiclassical approximation*, which consists in describing the internal degrees of freedom of the atom quantum-mechanically, and its external degrees of freedom (position $\mathbf{r}$ and momentum $\mathbf{p}$ of the center of mass of the atom) classically. Consider the simple case of a “two-level” atom, with a ground level $|g\rangle$ and an excited level $|e\rangle$, separated by the energy $\hbar\omega_0$. Suppose that the light field is monochromatic, with angular frequency ω , quasi-resonant with the atom

$$|\Delta| \ll \omega_0 \quad \text{with the detuning } \Delta = \omega - \omega_0. \quad (\text{III.4})$$

For simplicity, let us restrict ourselves to an atom at rest¹ at the point $\mathbf{r}$. One can use the rotating-wave approximation and write the Hamiltonian describing the atom-field coupling in matrix form in the basis $\{|g\rangle, |e\rangle\}$:

$$\hat{H}(\mathbf{r}) = \frac{\hbar}{2} \begin{pmatrix} \Delta & \kappa^*(\mathbf{r}) \\ \kappa(\mathbf{r}) & -\Delta \end{pmatrix}, \quad (\text{III.5})$$

where $\kappa(\mathbf{r})$ is the Rabi frequency² characterizing the coupling of the atom with the light field at the point $\mathbf{r}$, proportional to the electric field at that point for an electric-dipole coupling. The role of the parameter λ is then played by the position $\mathbf{r}$ of the center of mass of the atom.

¹or moving slowly enough that the Doppler effect plays a negligible role.

²Note that the rotating-wave approximation requires the inequality $|\kappa| \ll \omega_0$, which is in practice very well satisfied in all experiments performed with cold atoms.

1-2 Evolution of the system

Consider the case where the parameter λ depends on time. At each instant t , we decompose the state vector $|\psi(t)\rangle$ on the basis of the instantaneous eigenstates of the Hamiltonian $\hat{H}[\lambda(t)]$:

$$|\psi(t)\rangle = \sum_n c_n(t) |\psi_n[\lambda(t)]\rangle. \quad (\text{III.6})$$

Let us substitute this expression into the Schrödinger equation

$$i\hbar \frac{d|\psi\rangle}{dt} = \hat{H}[\lambda(t)] |\psi(t)\rangle. \quad (\text{III.7})$$

The derivative of the left-hand side of this equation involves:

$$\frac{d}{dt} [c_n |\psi_n(\lambda)\rangle] = \dot{c}_n |\psi_n(\lambda)\rangle + c_n \frac{d|\psi_n(\lambda)\rangle}{dt} \quad \text{with} \quad \frac{d|\psi_n(\lambda)\rangle}{dt} = \dot{\lambda} \cdot |\nabla \psi_n(\lambda)\rangle \quad (\text{III.8})$$

where we have used the compact notation for the gradient of an eigenstate with respect to the components of the vector λ :

$$|\nabla \psi_n(\lambda)\rangle \equiv \nabla_{\lambda} [|\psi_n(\lambda)\rangle]. \quad (\text{III.9})$$

We arrive at the system of differential equations satisfied by the $c_n(t)$:

$$i\hbar \dot{c}_n = E_n(t) c_n(t) - \hbar \sum_{\ell} \alpha_{n,\ell}(t) c_{\ell}(t), \quad (\text{III.10})$$

where we have set

$$\hbar \alpha_{n,\ell}(t) = i\hbar \langle \psi_n | \left(\frac{d}{dt} |\psi_{\ell}\rangle \right) = i\hbar \dot{\lambda} \cdot \langle \psi_n(\lambda) | \nabla \psi_{\ell}(\lambda) \rangle. \quad (\text{III.11})$$

The quantity $\alpha_{n,\ell}(t)$ characterizes the rate at which the eigenvectors of $\hat{H}(\lambda)$ “rotate.” It can be expressed in terms of the time derivative of the Hamiltonian itself. Indeed, the eigenvalue equation (III.2) leads to:

$$\frac{d\hat{H}}{dt} |\psi_{\ell}\rangle + \hat{H} \left(\frac{d}{dt} |\psi_{\ell}\rangle \right) = \frac{dE_{\ell}}{dt} |\psi_{\ell}\rangle + E_{\ell} \left(\frac{d}{dt} |\psi_{\ell}\rangle \right) \quad (\text{III.12})$$

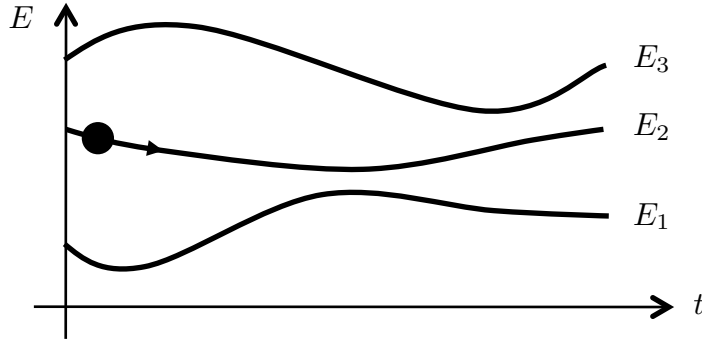


Figure III.1. Time variation of the energies E_n associated with the eigenstates $|\psi_n\rangle$ of the Hamiltonian $\hat{H}[\lambda]$.

i.e., multiplying by $\langle\psi_n|$ for $n \neq \ell$:

$$\langle\psi_n|\frac{d\hat{H}}{dt}|\psi_\ell\rangle = (E_\ell - E_n)\langle\psi_n|\left(\frac{d}{dt}|\psi_\ell\rangle\right) = i\hbar\omega_{n,\ell}\alpha_{n,\ell}. \quad (\text{III.13})$$

where we have introduced the instantaneous Bohr angular frequency between states n and ℓ

$$\hbar\omega_{n,\ell}(t) = E_n[\lambda(t)] - E_\ell[\lambda(t)]. \quad (\text{III.14})$$

1-3 Principle of the adiabatic approximation

In this paragraph, we follow the approach of the book by Messiah (2003). To simplify the analysis, let us begin by directly accounting for the evolution associated with the energy term in the system of differential equations (III.10). To do this, we introduce the coefficients $\tilde{c}_n(t)$

$$\tilde{c}_n(t) = e^{i\int_0^t E_n(t') dt'/\hbar} c_n(t), \quad (\text{III.15})$$

so that the system of differential equations (III.10) can be rewritten as

$$i\dot{\tilde{c}}_n = -\sum_\ell e^{i\int_0^t \omega_{n,\ell}(t') dt'} \alpha_{n,\ell}(t) \tilde{c}_\ell(t). \quad (\text{III.16})$$

Suppose that the initial state of the system $|\psi(0)\rangle$ is one of the eigenstates $|\psi_\ell\rangle$ of the Hamiltonian (cf. figure V.4)

$$c_\ell(0) = 1, \quad c_n(0) = 0 \quad \text{if } n \neq \ell. \quad (\text{III.17})$$

If the parameter λ is constant, the coefficients $\alpha_{n,\ell}$ vanish, and the solution of the system (III.16) is immediate:

$$\dot{\lambda} = 0 \Rightarrow \tilde{c}_n(t) = \delta_{n,\ell}. \quad (\text{III.18})$$

If λ evolves slowly in time, one can look for a solution of the differential system as a power series in $\dot{\lambda}$, with, to lowest order, for the coefficients $n \neq \ell$:

$$i\dot{\tilde{c}}_n = -e^{i\int_0^t \omega_{n,\ell}(t') dt'} \alpha_{n,\ell}(t), \quad \tilde{c}_n(0) = 0. \quad (\text{III.19})$$

To estimate the evolution of the probability amplitudes c_n for the states ψ_n such that $n \neq \ell$, let us first consider the simple case where neither the Bohr angular frequency $\omega_{n,\ell}$ nor the “rotation rate” $\alpha_{n,\ell}$ depends on time. In this case, the solution of (III.19) is very simple:

$$\tilde{c}_n(t) = \frac{1}{\omega_{n,\ell}} (e^{i\omega_{n,\ell}t} - 1) \alpha_{n,\ell}. \quad (\text{III.20})$$

In particular, if the parameter λ evolves “slowly,” the coefficient $\alpha_{n,\ell}$ is small, and the amplitude c_n will remain small at all times:

$$n \neq \ell : \quad \frac{|\alpha_{n,\ell}|}{|\omega_{n,\ell}|} \ll 1 \Rightarrow |c_n(t)| \ll 1 \quad \forall t. \quad (\text{III.21})$$

In the general case, the quantities $\omega_{n,\ell}$ and $\alpha_{n,\ell}$ vary in time, but one can impose the following sufficient condition for the population of each state ψ_n , with $n \neq \ell$, to remain small:

$$\left| \frac{\alpha_\ell^{\max}}{\omega_\ell^{\min.}} \right| = \frac{\text{maximum angular velocity of } \psi_\ell}{\text{minimum Bohr angular frequency associated with } \psi_\ell} \ll 1. \quad (\text{III.22})$$

Note that this condition excludes the case where the initially occupied state ψ_ℓ momentarily becomes degenerate with another state ψ_n .

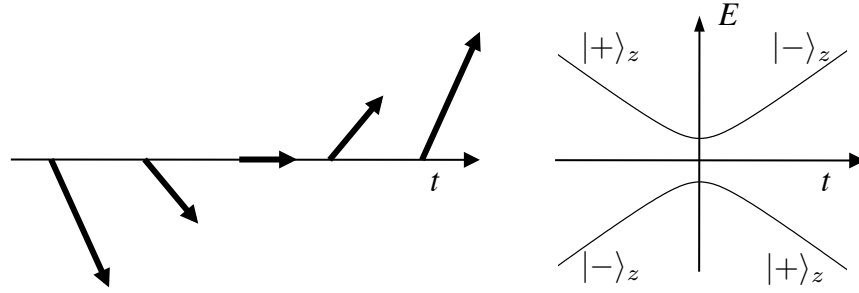


Figure III.2. Left: switching of a magnetic field according to law (III.24). Right: energies of the eigenstates of $\hat{H}(t)$ given in (III.25) for $\mu B_0 > 0$. These energies form the two branches of a hyperbola (cf. III.26).

When condition (III.22) is satisfied, one can thus assume, to a good approximation³, that the system adiabatically follows the state ψ_ℓ , i.e., that it remains, with a probability close to 1, in this state ψ_ℓ :

$$|c_\ell(t)| \approx 1 \quad \forall t. \quad (\text{III.23})$$

The question that remains to be studied is that of the phase of the coefficient $c_\ell(t)$, which will be the subject of the following section.

1-4 Example

The validity condition (III.22) takes a remarkably simple form if we consider the example already mentioned above of a magnetic moment μ placed in a time-dependent magnetic field. Consider a field $\mathbf{B}(t)$ varying according to (cf. figure III.2):

$$\mathbf{B}(t) = B_0 (\mathbf{u}_x + (t/\tau) \mathbf{u}_z). \quad (\text{III.24})$$

This field thus points in the directions $\pm \mathbf{u}_z$ at $t = \pm\infty$ and switches, over a time $\sim 2\tau$, between these two directions, being parallel to $\mathbf{u}_x$ at

³There is, of course, a certain amount of optimism in this statement. If the dimension of the Hilbert space is large, or even infinite, one can easily imagine situations where the population of each individual state ψ_n remains small, but the population of the initial state ψ_ℓ falls noticeably below 1.

the intermediate time $t = 0$. Suppose that the magnetic moment μ is associated with a spin 1/2, so that the Hamiltonian is written, in the basis $\{|+\rangle_z, |-\rangle_z\}$, as:

$$\hat{H}(t) = -\hat{\boldsymbol{\mu}} \cdot \mathbf{B}(t) = -\mu B_0 \begin{pmatrix} t/\tau & 1 \\ 1 & -t/\tau \end{pmatrix}, \quad \mu > 0. \quad (\text{III.25})$$

The two eigenenergies vary in time, forming the two branches of a hyperbola (figure III.2):

$$E_\pm(t) = \pm \mu B_0 (1 + t^2/\tau^2)^{1/2} \quad (\text{III.26})$$

Consider the case where the spin is prepared in the state $|-\rangle_z$ at $t = -\infty$ (lower branch of the hyperbola). Let us find the probability that this magnetic moment remains on the state corresponding to this branch of the hyperbola over time, ending up consequently in the state $|+\rangle_z$ at $t = +\infty$. It is simple to evaluate the criterion (III.22) in this case. The minimum Bohr angular frequency (here the Larmor angular frequency) is obtained at time $t = 0$ and equals $\omega_0 = 2\mu B_0/\hbar$. The rotation rate of the eigenaxes of $\hat{H}$ is maximal at the same point and equals $1/(2\tau)$, according to (III.13). The criterion (III.22) is thus written, up to a multiplicative factor,

$$\omega_0 \tau \gg 1. \quad (\text{III.27})$$

For the adiabatic approximation to be valid, the magnetic moment must undergo many rotations during the time τ , characteristic of the switching time of the magnetic field.

2 The Berry phase

In this paragraph we will follow an approach very close to the one presented by Michael Berry in his founding article (Berry 1984). Let us note, however, that precursors of this geometric phase had appeared before Berry's work in various branches of physics:

- In optics, with the work of Pancharatnam (1956). The latter had studied the phase change of polarized light with a given wave vector when the light beam passes through a series of polarizers, the polarization returning to its initial state at the output of the optical system.

- In molecular physics, with the analysis by Mead & Truhlar (1979) of certain subtleties in the Born–Oppenheimer approximation. Mead had in particular introduced the notion of the *molecular Aharonov–Bohm effect* in 1980.

The history of the development of these various concepts is retraced in the articles of Berry (1989) and Mead (1992), with numerous references to articles that had “anticipated” the existence of the geometric phase.

2-1 Dynamical phase and geometric phase

Here we consider a quantum system equivalent to the one in the previous section, and we assume that the adiabatic approximation holds. The system is prepared at the initial time $t = 0$ in an eigenstate ψ_ℓ of the Hamiltonian $\hat{H}[\lambda(0)]$, and we assume that it remains in this state at every instant

$$|\psi(t)\rangle = \sum_n c_n(t) |\psi_n[\lambda(t)]\rangle \approx c_\ell(t) |\psi_\ell[\lambda(t)]\rangle. \quad (\text{III.28})$$

The coefficient c_ℓ is a solution of the equation deduced from (III.10)

$$\begin{aligned} i\hbar \dot{c}_\ell &= \left[E_\ell(t) - i\hbar \dot{\lambda} \cdot \langle \psi_\ell | \nabla \psi_\ell \rangle \right] c_\ell, \\ &= \left[E_\ell(t) - \dot{\lambda} \cdot \mathcal{A}_\ell(\lambda) \right] c_\ell, \end{aligned} \quad (\text{III.29})$$

where we have introduced the real vector⁴ called the *Berry connection*

$$\mathcal{A}_\ell(\lambda) = i\hbar \langle \psi_\ell | \nabla \psi_\ell \rangle, \quad (\text{III.31})$$

This Berry connection will play the role of a vector potential in what follows.

The evolution equation (III.29) for the amplitude c_ℓ can be simply integrated to give:

$$c_\ell(t) = e^{i\Phi^{\text{dyn.}}(t)} e^{i\Phi^{\text{geom.}}(t)} c_\ell(0) \quad (\text{III.32})$$

⁴To show that $\mathcal{A}_\ell$ is real, let us use the fact that the vector $|\psi_\ell(\lambda)\rangle$ is normalized:

$$\langle \psi_\ell | \psi_\ell \rangle = 1 \Rightarrow \langle \nabla \psi_\ell | \psi_\ell \rangle + \langle \psi_\ell | \nabla \psi_\ell \rangle = 0 \Rightarrow \langle \psi_\ell | \nabla \psi_\ell \rangle \text{ imaginary.} \quad (\text{III.30})$$

where we have introduced *the dynamical phase*

$$\Phi^{\text{dyn.}}(t) = -\frac{1}{\hbar} \int_0^t E_\ell(t') dt' \quad (\text{III.33})$$

and *the geometric phase*

$$\Phi^{\text{geom.}}(t) = \frac{1}{\hbar} \int_0^t \dot{\lambda}(t') \cdot \mathcal{A}_\ell[\lambda(t')] dt' = \frac{1}{\hbar} \int_{\lambda(0)}^{\lambda(t)} \mathcal{A}_\ell(\lambda) \cdot d\lambda. \quad (\text{III.34})$$

The last integral is taken along the line $\lambda(t')$, for t' ranging from 0 to t .

This distinction between the two phases is fairly natural:

1. The dynamical phase is the usual phase that appears even for a time-independent system; it is responsible for the “motion” in quantum mechanics, when a given system is prepared in a superposition of eigenstates of different energies.
2. The geometric phase depends only on the “trajectory” followed by the parameter λ during the evolution. It does not depend on the time taken to go from $\lambda(0)$ to $\lambda(t)$, provided this time is long enough to ensure the validity of the adiabatic approximation.

2-2 The case of a closed circuit

An important special case of the preceding analysis is that of a closed loop $\mathcal{C}$ for the parameter λ :

$$\lambda(0) \longrightarrow \lambda(t) \longrightarrow \lambda(T) = \lambda(0). \quad (\text{III.35})$$

The geometric phase is then given by the integral of the Berry connection over this loop

$$\Phi^{\text{geom.}}(\mathcal{C}) = \frac{1}{\hbar} \oint_{\mathcal{C}} \mathcal{A}_\ell(\lambda) \cdot d\lambda, \quad (\text{III.36})$$

an expression that is obviously tempting to compare with the Aharonov–Bohm phase encountered in a previous chapter. Exploring this analogy further will occupy us for the rest of this course.

Let us immediately point out the following essential property: the geometric phase $\Phi^{\text{geom.}}(\mathcal{C})$ obtained for a closed circuit is *independent* of the choice of gauge used to define the eigenstates $|\psi_n\rangle$. Suppose that we perform the transformation

$$|\psi_n(\boldsymbol{\lambda})\rangle \longrightarrow |\tilde{\psi}_n(\boldsymbol{\lambda})\rangle = e^{i\Omega(\boldsymbol{\lambda})/\hbar} |\psi_n(\boldsymbol{\lambda})\rangle. \quad (\text{III.37})$$

The Berry connection is then modified as follows:

$$\tilde{\mathcal{A}}_\ell(\boldsymbol{\lambda}) = \mathcal{A}_\ell(\boldsymbol{\lambda}) - \nabla\Omega(\boldsymbol{\lambda}) \quad (\text{III.38})$$

In an integral over a closed circuit, the contribution of the additional term

$$- \oint \nabla\Omega(\boldsymbol{\lambda}) \cdot d\boldsymbol{\lambda} \quad (\text{III.39})$$

vanishes, which ensures the gauge invariance of $\Phi^{\text{geom.}}(\mathcal{C})$. This geometric phase is thus an intrinsic physical quantity of the system and of its history, which answers the question

“Which way did the system go on its journey?”

whereas the dynamical phase (also gauge invariant, since it depends only on the eigenvalues, not on the eigenstates) provides the answer to the other question:

“How long did this journey take?”

2-3 Berry curvature and artificial magnetic field

From now on we will assume that the parameter $\boldsymbol{\lambda}$ evolves in a three-dimensional space:

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3). \quad (\text{III.40})$$

This is in particular the case when this parameter represents the position of the center of mass of an atom or the quasi-momentum of a particle in a periodic potential. One can then use the Kelvin–Stokes formula to transform the line integral (III.36) into an integral over a surface S bounded by the contour $\mathcal{C}$:

$$\Phi^{\text{geom.}}(\mathcal{C}) = \frac{1}{\hbar} \iint_S \mathcal{B}_\ell \cdot d^2S \quad (\text{III.41})$$

where we have introduced the *Berry curvature*:

$$\mathcal{B}_\ell = \nabla \times \mathcal{A}_\ell. \quad (\text{III.42})$$

The calculation of the curl proceeds without difficulty

$$\mathcal{B}_\ell = i\hbar \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \times \begin{pmatrix} \langle \psi_\ell | \partial_1 \psi_\ell \rangle \\ \langle \psi_\ell | \partial_2 \psi_\ell \rangle \\ \langle \psi_\ell | \partial_3 \psi_\ell \rangle \end{pmatrix} = i\hbar \begin{pmatrix} \langle \partial_2 \psi_\ell | \partial_3 \psi_\ell \rangle - \langle \partial_3 \psi_\ell | \partial_2 \psi_\ell \rangle \\ \dots \\ \dots \end{pmatrix} \quad (\text{III.43})$$

with $\partial_i = \frac{\partial}{\partial \lambda_i}$, which can also be written in compact form:

$$\mathcal{B}_\ell = i\hbar \langle \nabla \psi_\ell | \times | \nabla \psi_\ell \rangle, \quad (\text{III.44})$$

where $\mathcal{B}_\ell$ is a real quantity, whose analogy with a magnetic field one can already sense. To explore this point further, we will examine the influence of a quantum gauge transformation on this quantity.

2-4 Influence of a gauge transformation

We have already noted that under the transformation

$$|\psi_n(\boldsymbol{\lambda})\rangle \longrightarrow |\tilde{\psi}_n(\boldsymbol{\lambda})\rangle = e^{i\Omega(\boldsymbol{\lambda})/\hbar} |\psi_n(\boldsymbol{\lambda})\rangle.$$

the Berry connection was modified as follows

$$\mathcal{A}_\ell(\boldsymbol{\lambda}) \longrightarrow \tilde{\mathcal{A}}_\ell(\boldsymbol{\lambda}) = \mathcal{A}_\ell(\boldsymbol{\lambda}) - \nabla\Omega.$$

Taking the curl of this equation, it is then immediate to deduce

$$\mathcal{B}_\ell(\boldsymbol{\lambda}) \longrightarrow \tilde{\mathcal{B}}_\ell(\boldsymbol{\lambda}) = \mathcal{B}_\ell(\boldsymbol{\lambda}) \quad (\text{III.45})$$

Since the Berry curvature $\mathcal{B}_\ell(\boldsymbol{\lambda})$ is gauge invariant, it can be regarded as a physical quantity of the system, just like the Berry phase attached to a closed circuit $\mathcal{C}$. However, unlike the geometric phase, which characterizes a trajectory in the space of $\boldsymbol{\lambda}$, the Berry curvature is a local quantity, just like the magnetic field in classical electromagnetism.

The results found for $\Phi^{\text{geom.}}$, $\mathcal{A}_\ell$, and $\mathcal{B}_\ell$ thus show a deep similarity with those obtained for a charge moving in a magnetic field. In the particular case where the parameter $\boldsymbol{\lambda}$ denotes the position $\mathbf{r}$ of the particle, we have the direct correspondence

$$\boldsymbol{\lambda} \longleftrightarrow \mathbf{r}, \quad \mathcal{A}_\ell \longleftrightarrow q\mathbf{A}, \quad \mathcal{B}_\ell \longleftrightarrow q\mathbf{B}. \quad (\text{III.46})$$

and the Berry connection $\mathcal{A}_\ell$ has, in this case, the dimension of a momentum.

For explicit calculations, it can be useful to have an expression for $\mathcal{B}_\ell$ in which gauge invariance is manifest. To this end, let us start again from expression (III.43) and insert a closure relation into the various terms:

$$\langle \partial_2 \psi_\ell | \partial_3 \psi_\ell \rangle - \langle \partial_3 \psi_\ell | \partial_2 \psi_\ell \rangle = \sum_n \langle \partial_2 \psi_\ell | \psi_n \rangle \langle \psi_n | \partial_3 \psi_\ell \rangle - \langle \partial_3 \psi_\ell | \psi_n \rangle \langle \psi_n | \partial_2 \psi_\ell \rangle. \quad (\text{III.47})$$

Note that the term $n = \ell$ of the sum has a zero contribution, because of the relation

$$\langle \psi_\ell | \psi_\ell \rangle = 1 \Rightarrow \langle \partial_j \psi_\ell | \psi_\ell \rangle + \langle \psi_\ell | \partial_j \psi_\ell \rangle = 0. \quad (\text{III.48})$$

Moreover, the eigenvalue equation $\hat{H}|\psi_\ell\rangle = E_\ell|\psi_\ell\rangle$ gives, after differentiation:

$$(\partial_j \hat{H})|\psi_\ell\rangle + \hat{H}|\partial_j \psi_\ell\rangle = (\partial_j E_\ell)|\psi_\ell\rangle + E_\ell|\partial_j \psi_\ell\rangle \quad (\text{III.49})$$

i.e., for $n \neq \ell$:

$$\langle \psi_n | \partial_j \hat{H} | \psi_\ell \rangle = (E_\ell - E_n) \langle \psi_n | \partial_j \psi_\ell \rangle. \quad (\text{III.50})$$

Substituting this expression into (III.47), and then into (III.43), we arrive at the following expression for the Berry curvature:

$$\mathcal{B}_\ell(\lambda) = i\hbar \sum_{n \neq \ell} \frac{\langle \psi_\ell | \nabla \hat{H} | \psi_n \rangle \times \langle \psi_n | \nabla \hat{H} | \psi_\ell \rangle}{(E_\ell - E_n)^2}. \quad (\text{III.51})$$

In this form, gauge invariance is manifest, since each term of the sum is unchanged when the phase of one or more eigenstates $|\psi_n\rangle$ is modified. This expression is well suited to numerical calculations, in which it is not always easy to impose a regular phase on the various eigenstates resulting from the diagonalization of $\hat{H}$.

Remark 1. We have assumed throughout the above that the adiabatic approximation is valid, which requires that the state ψ_ℓ never have the same energy as another state ψ_n . One sees more precisely from (III.51) that if the energy of the state ψ_ℓ approaches that of a state ψ_n , the corresponding term of the sum will become very large, and may even diverge if there is a point λ where the two energies are equal.

Remark 2. We have defined a Berry curvature for each eigenstate ψ_ℓ of the Hamiltonian. Let us now consider the quantity

$$\sum_\ell \mathcal{B}_\ell(\lambda). \quad (\text{III.52})$$

It is clear from (III.51) that this quantity always vanishes. Indeed, each pair of eigenstates (ψ_ℓ, ψ_n) contributes twice to the sum, these two contributions being opposite.

3 Examples

3-1 Spin in an external magnetic field

One of the simplest examples in which the Berry phase appears concerns the case already mentioned above of a magnetic moment $\hat{\mu}$ associated with a spin $S = 1/2$:

$$\hat{\mu} = \mu \hat{\sigma}, \quad (\text{III.53})$$

where the $\hat{\sigma}_i$, $i = x, y, z$, are the three Pauli matrices:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{III.54})$$

This magnetic moment is placed in an external magnetic field $\mathbf{B}$. We have already introduced the coupling Hamiltonian

$$\hat{H} = -\mathbf{B} \cdot \hat{\mu} = \frac{\hbar\omega}{2} \mathbf{n} \cdot \hat{\sigma} \quad (\text{III.55})$$

where the Larmor angular frequency ω is defined by

$$\frac{\hbar\omega}{2} = -\mu B, \quad (\text{III.56})$$

where $\mathbf{n}$ is the unit vector giving the direction of the magnetic field ($\mathbf{n} = \mathbf{B}/B$). The eigenstates of the Hamiltonian are denoted $|\pm : \mathbf{n}\rangle$ and have energy $\pm\hbar\omega/2$. In this problem, the external field plays the role of the parameter λ , so that

$$\lambda \equiv B, \quad \nabla H = -\hat{\mu}. \quad (\text{III.57})$$

The calculation of the Berry curvature defined in (III.51) is then simple, since only a single term appears in the sum over $n \neq \ell$:

$$\mathcal{B}_+ = i\hbar \frac{\langle + : \mathbf{n} | \hat{\boldsymbol{\mu}} | - : \mathbf{n} \rangle \times \langle - : \mathbf{n} | \hat{\boldsymbol{\mu}} | + : \mathbf{n} \rangle}{(\hbar\omega)^2}, \quad \mathcal{B}_- = -\mathcal{B}_+. \quad (\text{III.58})$$

The calculation of the numerator of this expression is somewhat technical, but poses no difficulty of principle. To simplify it, let us assume that $\mathbf{n}$ is parallel to the z axis. We then have:

$$\langle + : \mathbf{u}_z | \hat{\boldsymbol{\mu}} | - : \mathbf{u}_z \rangle = \mu(\mathbf{u}_x + i\mathbf{u}_y), \quad (\text{III.59})$$

so that the numerator is equal to $\mu^2 (\mathbf{u}_x + i\mathbf{u}_y) \times (\mathbf{u}_x - i\mathbf{u}_y) = -i2\mu^2 \mathbf{u}_z$. Returning to an arbitrary orientation for $\mathbf{B}$, we find:

$$\mathcal{B}_+ = +\hbar \frac{\mathbf{B}}{2B^3}, \quad \mathcal{B}_- = -\hbar \frac{\mathbf{B}}{2B^3}. \quad (\text{III.60})$$

This formula makes it possible to directly evaluate the Berry phase acquired by the magnetic moment when the magnetic field $\mathbf{B}$ slowly traces out a closed contour⁵

$$\mathcal{C} : \quad \mathbf{B}(0) \rightarrow \mathbf{B}(t) \rightarrow \mathbf{B}(T) = \mathbf{B}(0). \quad (\text{III.61})$$

The general result (III.41) is written, in this particular case,

$$\Phi_{\pm}^{\text{geom.}}(\mathcal{C}) = \frac{1}{\hbar} \iint_S \mathcal{B}_{\pm} \cdot d^2\mathbf{S} = \pm \iint_S \frac{\mathbf{B}}{2B^3} \cdot d^2\mathbf{S} = \pm \frac{\Omega}{2}, \quad (\text{III.62})$$

where the surface S is bounded by the contour $\mathcal{C}$, and where Ω denotes the solid angle subtended by $\mathcal{C}$ as seen from the origin $\mathbf{B} = 0$.

The experiment was carried out shortly after Berry's proposal, on a neutron beam, by Bitter & Dubbers (1987). Neutrons with velocity $v = 500$ m/s propagate inside a 40 cm long solenoid, within which there is a helical magnetic field, with a uniform component along z and a component rotating in the xy plane (figure III.3). The winding is such that the field $\mathbf{B}$ makes

⁵Let us recall once again the following assumption: there is no degeneracy between the energy levels of $\hat{H}$, which in this case means that the contour does not pass through the zero field $\mathbf{B} = 0$. If the contour did pass through this point, it would not be possible to use the adiabatic approximation, and the result (III.60) would lose its meaning.

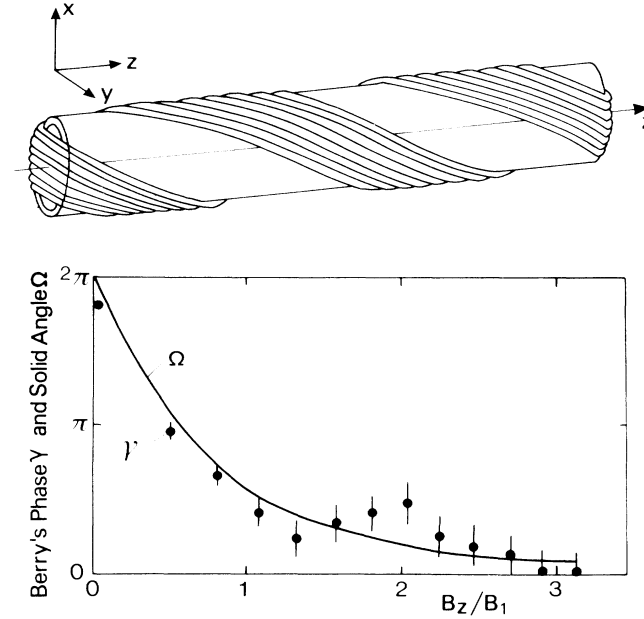


Figure III.3. Top: solenoid used by Bitter & Dubbers (1987) to demonstrate the Berry phase on the spin of a neutron beam propagating along the z direction. Bottom: comparison between the Berry phase accumulated by a neutron and the solid angle traced out by the magnetic field along the trajectory.

one complete turn over the length of the solenoid: this is indeed a closed trajectory in parameter space. The polarization state of the neutrons at the entrance of the solenoid is adjustable and can be decomposed on the two local eigenstates $|\pm\rangle$. The difference between the phases accumulated by each component results in a rotation of the polarization at the output of the solenoid. The experiment gave results in excellent agreement with Berry's prediction: the accumulated phase is proportional to the solid angle traced out by the magnetic field $\mathbf{B}$ along the trajectory of the neutrons. This experiment was later repeated with neutral atoms by Miniatura, Robert, et al. (1992).

3-2 Connection with parallel transport

We obtained the expression for the Berry phase using quantum concepts: the Schrödinger equation, the phase of a state vector, etc. Nevertheless, equivalent notions appear in classical physics (Hannay 1985); we will give an example based on the notion of parallel transport.

Consider a sphere centered at $\mathbf{r} = 0$, for example the surface of the Earth, and let us give ourselves a vector $\mathbf{e}_i$ at a given initial point $\mathbf{r}_i$ on the sphere, specified by its latitude θ_i and its longitude φ_i . We define the notion of parallel transport of this vector to another point $\mathbf{r}$ on the sphere by imposing:

- that the transported vector remains tangent to the sphere, i.e., $\mathbf{e} \cdot \mathbf{r} = 0$ at every instant,
- that no torsion normal to the surface of the sphere is induced during the transport. Given the right-handed frame

$$\{\mathbf{e}, \mathbf{e}', \mathbf{r}\} \quad \text{with} \quad \mathbf{e}' = \mathbf{r} \times \mathbf{e}, \quad (\text{III.63})$$

the vector $\boldsymbol{\Omega}$ describing, at each instant, the rotation of this frame via

$$\dot{\mathbf{e}} = \boldsymbol{\Omega} \times \mathbf{e}, \quad \dot{\mathbf{r}} = \boldsymbol{\Omega} \times \mathbf{r}, \quad \dot{\mathbf{e}}' = \boldsymbol{\Omega} \times \mathbf{e}', \quad (\text{III.64})$$

must satisfy

$$\boldsymbol{\Omega} \cdot \mathbf{r} = 0. \quad (\text{III.65})$$

In other words, the evolution of the vectors $\mathbf{e}$ and $\mathbf{e}'$ must satisfy

$$\mathbf{e} \cdot \dot{\mathbf{e}}' = 0, \quad \dot{\mathbf{e}} \cdot \mathbf{e}' = 0. \quad (\text{III.66})$$

Suppose that we carry out a parallel transport along a closed circuit $\mathcal{C}$ on the surface of the sphere. Starting from a vector $\mathbf{e}_i$, the (nontrivial) calculation shows that the vector $\mathbf{e}_f$ obtained after transport along this circuit has rotated about $\mathbf{r}$ by an angle $\alpha(\mathcal{C})$, which is nothing other than the solid angle subtended by $\mathcal{C}$ as seen from the center of the sphere. The same rotation⁶ occurs for the vector $\mathbf{e}'_i$, which allows this parallel-transport result to

⁶This is necessarily the same, since we have imposed that the frame $\{\mathbf{e}, \mathbf{e}', \mathbf{r}\}$ remain orthogonal and right-handed during the transport.

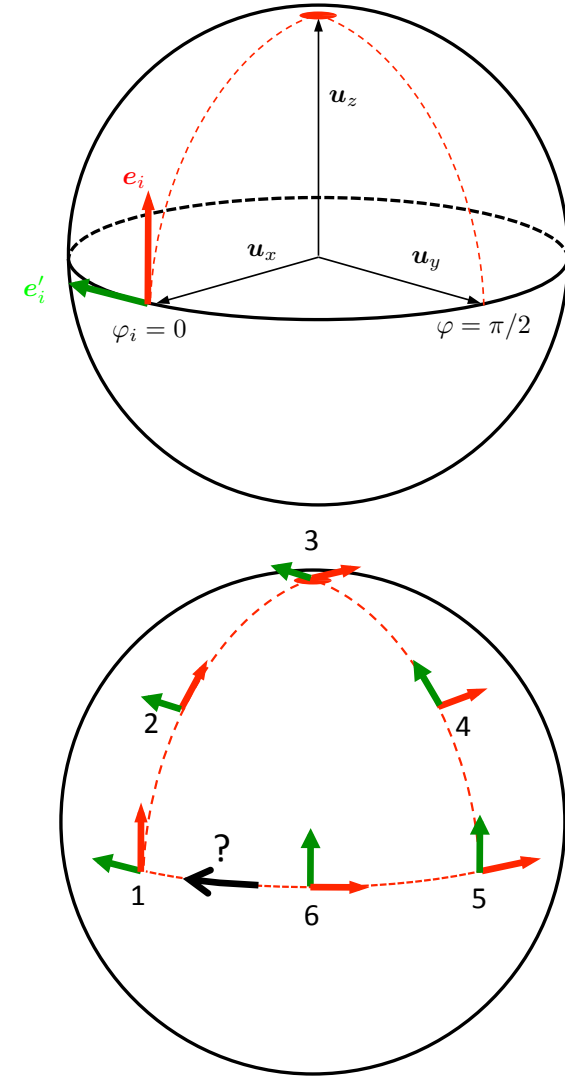


Figure III.4. Parallel transport: the top panel shows the initial situation presented in equation (III.68). The bottom drawing shows 6 steps of the transport described in the text. When the loop is completed after step 6, the pair of vectors $\mathbf{e}, \mathbf{e}'$ no longer coincides with the initial vectors $\mathbf{e}_i, \mathbf{e}'_i$.

be cast in a form very close to Berry's geometric phase:

$$\begin{aligned} \mathcal{C}: \quad \Psi_i = e_i + i e'_i &\longrightarrow \Psi_f = e_f + i e'_f \\ &= (e_i \cos \alpha + e'_i \sin \alpha) + i(-e_i \sin \alpha + e'_i \cos \alpha) \\ &= e^{-i\alpha} \Psi_i. \end{aligned} \quad (\text{III.67})$$

We will not prove this relation in the general case (see for example Berry (1989)), but let us verify it on an example (figure III.4). Let us start from the point r located on the equator with longitude $\varphi_i = 0$, and take e_i pointing toward the pole and e'_i aligned with the equator:

$$\text{Start on the equator, } \varphi = 0: \quad e_i = u_z, \quad e'_i = -u_y. \quad (\text{III.68})$$

Let us follow the path going up along the meridian from the equator to the North Pole. We then have:

$$\text{North Pole:} \quad e = -u_x, \quad e' = -u_y. \quad (\text{III.69})$$

Once at the pole, let us come back down along another meridian, specifically the one at longitude $\varphi = \pi/2$. We see that, once we arrive at the equator, the vectors e, e' have been transported to

$$\text{Equator, } \varphi = \pi/2: \quad e = -u_x, \quad e' = +u_z. \quad (\text{III.70})$$

To close the circuit, it remains to perform a quarter turn on the equator, which does not change e' and sends e to u_y :

$$\text{End on the equator, } \varphi = 0: \quad e_f = u_y, \quad e'_f = u_z. \quad (\text{III.71})$$

The two vectors e and e' have thus rotated by an angle $\pi/2$ during this parallel transport along the contour $\mathcal{C}$. This result agrees with the general formula (III.67), since this closed contour subtends a (negative) solid angle whose absolute value equals $1/8$ of 4π , i.e., $\pi/2$:

$$\Psi_i = u_z - i u_y \rightarrow \Psi_f = u_y + i u_z = e^{i\pi/2} \Psi_i. \quad (\text{III.72})$$

3-3 The Foucault pendulum

A famous example of parallel transport, and hence of geometric phase, is the Foucault pendulum. Recall that in Foucault's experiment, one considers a pendulum suspended from a point fixed in the Earth's frame, and

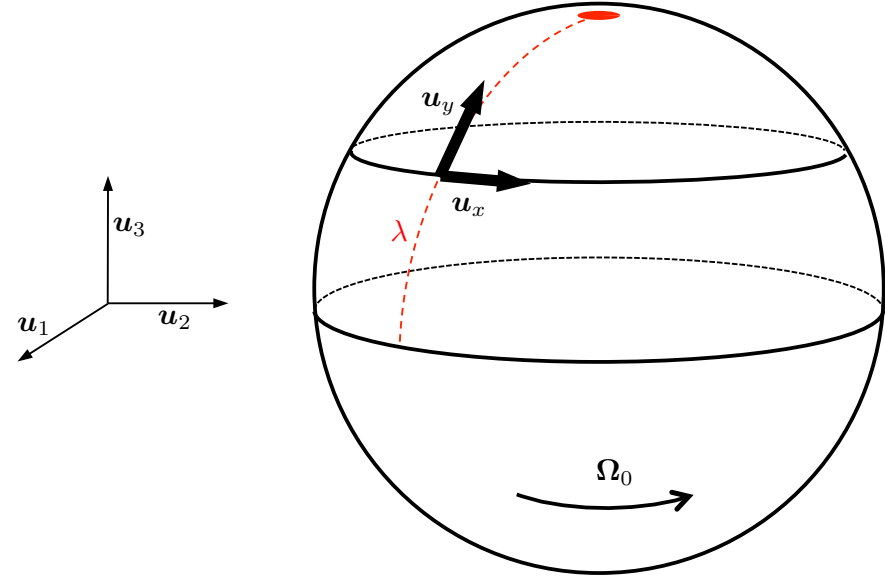


Figure III.5. Locating a point on the surface of the Earth for the analysis of the motion of the Foucault pendulum.

one observes the oscillation plane of the pendulum rotate with an angular velocity

$$\Omega = \Omega_0 \sin \lambda \quad (\text{III.73})$$

where Ω is the angular velocity of Earth's rotation about its axis [$1/(24 \text{ hours})$] and λ is the latitude of the place. On the equator, there is no precession of the oscillation plane; at the poles, the precession takes 24 hours⁷. The precession is clockwise in the northern hemisphere and counterclockwise in the southern hemisphere.

Let us recall the mathematical treatment of the oscillation of the pendulum. We start by choosing a fixed frame u_1, u_2, u_3 (cf. figure III.5), and then a frame attached to the Earth, u_x, u_y, u_z , with the vertical oriented

⁷The phenomenon is then trivial, since the point where the pendulum is attached does not move.

along $\mathbf{u}_z$. In the horizontal plane, we can take, for example, $\mathbf{u}_x$ pointing east and $\mathbf{u}_y$ pointing north

$$\mathbf{u}_x = -\mathbf{u}_1 \sin(\Omega_0 t) + \mathbf{u}_2 \cos(\Omega_0 t), \quad (\text{III.74})$$

$$\mathbf{u}_y = -[\mathbf{u}_1 \cos(\Omega_0 t) + \mathbf{u}_2 \sin(\Omega_0 t)] \sin \lambda + \mathbf{u}_3 \cos \lambda \quad (\text{III.75})$$

The oscillation of the pendulum takes place almost entirely in the horizontal plane, and the equation of motion in this plane is written, to a good approximation:

$$\ddot{x} = -\omega^2 x + 2\Omega \dot{y}, \quad \ddot{y} = -\omega^2 y - 2\Omega \dot{x}, \quad (\text{III.76})$$

where ω is the angular frequency of the pendulum and where the term proportional to Ω is due to the Coriolis force. In the (realistic) regime where $\Omega \ll \omega$, these equations are solved by introducing the complex variable $u = x + iy$, which obeys the equation

$$\ddot{u} + 2i\Omega \dot{u} + \omega^2 u = 0 \quad (\text{III.77})$$

and one looks for an approximate solution of the form $u(t) = \tilde{u}(t) e^{i\omega t}$, where $\tilde{u}$ determines the orientation of the oscillation plane of the pendulum and evolves slowly: $\tilde{u}(t) \approx \tilde{u}(0) e^{-i\Omega t}$. More precisely, the oscillation occurs along the line with direction vector

$$\mathbf{U} = \text{Re}(\tilde{u}) \mathbf{u}_x + \text{Im}(\tilde{u}) \mathbf{u}_y. \quad (\text{III.78})$$

One can then check that the vector $\mathbf{U}$ determining the oscillation plane of the pendulum evolves by undergoing parallel transport. If we denote by $\mathbf{V}$ the vector in the horizontal plane orthogonal to $\mathbf{U}$, the preceding expressions lead to

$$\frac{d\mathbf{U}}{dt} \cdot \mathbf{V} = 0, \quad \mathbf{U} \cdot \frac{d\mathbf{V}}{dt} = 0. \quad (\text{III.79})$$

After one complete rotation of the Earth about itself, the vector $\mathbf{U}$ therefore does not return to its starting point: it rotates by an amount equal to the solid angle traced out by the vertical at the location of the pendulum, i.e., $2\pi(1 - \sin \lambda)$. This is precisely the result found by Foucault.

3-4 Light in an optical fiber

Another example in which the geometric phase manifests itself very clearly is the propagation of polarized light in an isotropic single-mode fiber. If the fiber is straight, the polarization does not rotate, and everything is simple. If the fiber is curved, but planar, the situation is again easy to analyze. The polarization axis perpendicular to the plane is preserved, and consequently the other polarization axis, contained in the plane, rotates with the fiber.

The situation is more subtle when the fiber undergoes torsion and is no longer planar. This is the case for the geometry considered by Tomita and Chiao, reproduced in figure III.6, where the fiber is wound helically around a cylinder (Tomita & Chiao 1986). The winding is such that the tangent vectors to the fiber at the input and output are equal.

Linear polarization is sent in at the fiber input, and a likewise linear polarization is detected at the fiber output. But the latter has rotated relative to the initial polarization, by an angle directly related to the Berry phase, i.e., the solid angle traced out by the vector tangent to the fiber along the winding (figure III.6). Tomita & Chiao (1986) verified that this rotation of the polarization depended on the solid angle, but not on the regularity of the winding on the cylinder.

Tomita & Chiao (1986) initially interpreted their results in quantum terms, based on a treatment similar to the one we presented above: the polarization corresponds to the spin of the photon and the experiment – even though it is carried out with a large number of photons prepared in a coherent state – reveals this quantum phase. Some authors, such as Haldane (1987), contested this quantum interpretation, insisting that it was simply an effect related to differential geometry (see also the reply by Chiao & Tomita (1987)). An instructive summary of this debate was given by Berry (1990), who emphasizes two points:

- One can show explicitly, starting from Maxwell's equations, that the propagation of light in a fiber follows the laws of parallel transport for the polarization of the light (Berry 1987). The connection with the classical geometric phase follows naturally from this.
- The question of whether this is a classical or a quantum phenomenon

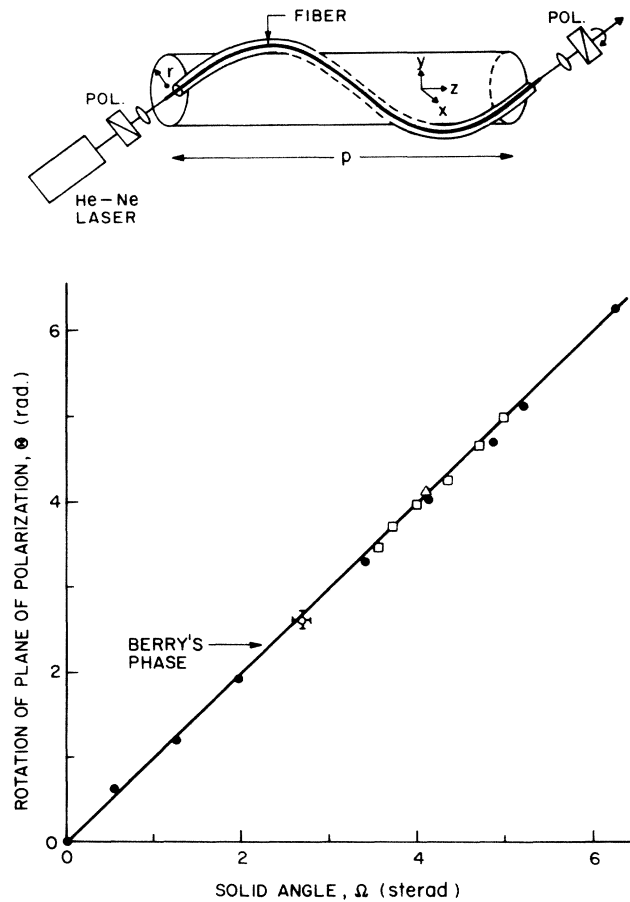


Figure III.6. Experiment of Tomita & Chiao (1986): linearly polarized light is sent into an optical fiber wound around a cylinder. At the output of the fiber, the light is still linearly polarized, but the polarization vector has rotated. The rotation angle is directly related to the solid angle swept out by the vector tangent to the fiber, as expected given the parallel-transport equations governing the propagation in the fiber. The different symbols correspond to different types of winding.

is, according to Berry, a “pseudo-problem,” and he supports this conclusion by quoting Feynman: *The photon equation is just the same as Maxwell’s equations.* In other words, for the processes at play here, the classical and quantum equations are the same, and the debate is moot.

4 Born–Oppenheimer-type approach

So far we have considered two types of variables to parametrize our system. On the one hand, we have described part of the dynamics of this system quantum-mechanically, the associated variables being, for example, the magnetic-moment operator $\hat{\mu}$ in (III.55). On the other hand, we have assumed the existence of variables external to the system, treated here as classical parameters and grouped under the generic name λ .

In the rest of this course we will be interested in situations where these variables λ are themselves quantum. The prototype example is the following: consider an atom that can move in a light field. One can be interested in a classical treatment of the motion of the atom’s center of mass, assigning it a position r (i.e. $\lambda \equiv r$). The evolution of the internal variables of the atom, i.e., the dynamics of its electronic structure, is then described by a Hamiltonian $\hat{H}_{\text{int}}(r)$ describing the coupling of the atom at point r with the light field. But one may also need a quantum description of the motion of the atoms’ center of mass, for example when the atoms are cold enough that the wave-like character of their motion plays an important role in the dynamics considered. The position r then becomes an operator, $\hat{r}$, conjugate to the momentum operator $\hat{p}$.

The question we must then address is the following: how do the previous results, such as the existence of a Berry curvature, translate to the case of a quantum description? We will see that this translation is in fact very direct, and that one arrives at a Hamiltonian in perfect analogy with that of a charged particle moving in a magnetic field (Berry 1989; Mead & Truhlar 1979; Moody, Shapere, et al. 1989): the Berry phase indeed provides a way of generating an artificial magnetic field.

4-1 Quantum treatment of the “slow variables”

In what follows we will consider a particle of mass M , with position and momentum operators $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}} = -i\hbar\nabla_{\mathbf{r}}$, endowed with an internal dynamics described by the Hamiltonian $\hat{H}_{\text{int}}(\mathbf{r})$. The dependence of this internal Hamiltonian on $\mathbf{r}$ arises from the fact that the particle may possess an electric or magnetic dipole (permanent or induced), and that this dipole can interact with externally imposed electromagnetic fields. The total Hamiltonian describing the motion of this particle is thus

$$\hat{H}_{\text{tot}} = \frac{\hat{\mathbf{p}}^2}{2M} \otimes \hat{1}_{\text{int}} + \hat{H}_{\text{int}}(\hat{\mathbf{r}}). \quad (\text{III.80})$$

The starting point of our treatment will be similar to what precedes. We assume that at every point $\mathbf{r}$ in space, we can diagonalize the internal Hamiltonian $\hat{H}_{\text{int}}$:

$$\hat{H}_{\text{int}}(\mathbf{r})|\psi_n(\mathbf{r})\rangle = E_n(\mathbf{r})|\psi_n(\mathbf{r})\rangle. \quad (\text{III.81})$$

At every point $\mathbf{r}$, the set $\{|\psi_n(\mathbf{r})\rangle\}$ forms an orthonormal basis of the Hilbert space associated with the internal degrees of freedom of the particle. In practice, this Hilbert space will be finite-dimensional: dimension 2 in the case of the simplest modeling of a “two-level atom,” dimension $(2J_g + 1) + (2J_e + 1)$ in the more realistic case of a transition between a ground level of angular momentum J_g coupled by a light field to an excited level of angular momentum J_e . Note that, for simplicity, we assume that the internal Hamiltonian $\hat{H}_{\text{int}}$ is time-independent. The eigenstates $|\psi_n(\mathbf{r})\rangle$ and the associated energies are thus also time-independent.

The *total* quantum state of the particle at a time t , characterizing both its internal and external degrees of freedom, can be written:

$$\Psi(\mathbf{r}, t) = \sum_n \phi_n(\mathbf{r}, t) |\psi_n(\mathbf{r})\rangle. \quad (\text{III.82})$$

Characterizing the dynamics of the atom then amounts to determining the probability amplitudes $\phi_n(\mathbf{r}, t)$ for finding the particle at point $\mathbf{r}$ in the internal state $|\psi_n(\mathbf{r})\rangle$.

Let us begin by substituting this expression into the Schrödinger equation:

tion:

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}_{\text{tot}} \Psi(\mathbf{r}, t) = \left(-\frac{\hbar^2}{2M} \Delta + \hat{H}_{\text{int}}(\mathbf{r}) \right) \Psi(\mathbf{r}, t) \quad (\text{III.83})$$

and let us project this equation onto a given internal state by multiplying it by $\langle \psi_n |$:

$$i\hbar \frac{\partial \phi_n}{\partial t} = -\frac{\hbar^2}{2M} \sum_{\ell} \langle \psi_n | [\Delta (\phi_{\ell} |\psi_{\ell}\rangle)] + E_n \phi_n \quad (\text{III.84})$$

where we have omitted the explicit dependence on $\mathbf{r}$ and t to simplify the notation. At this stage, no approximation has been made, and solving the system of partial differential equations (III.84) makes it possible, in principle, to extract the complete dynamics of the particle.

4-2 Effective Schrödinger equation

Suppose now that the particle is initially prepared in one of the internal states ψ_{ℓ} , and that the dynamics of this particle is such that one can assume, to a good approximation, that only this state ψ_{ℓ} remains appreciably occupied over time:

$$\forall t : \int |\phi_n(\mathbf{r}, t)|^2 d^3r \ll 1 \quad \text{if } n \neq \ell. \quad (\text{III.85})$$

This is the equivalent of the adiabatic approximation used above. The validity of this approximation depends both on the initial state $\phi_{\ell}(\mathbf{r}, t=0)$ (are the corresponding velocities low enough?) and on the structure of the Hamiltonian $\hat{H}_{\text{int}}$ (does the level ℓ not become nearly degenerate, are the possible accelerations not too large?). If this approximation is correct, which we will assume, we can keep only the term $n = \ell$ in the system of equations (III.84):

$$i\hbar \frac{\partial \phi_{\ell}}{\partial t} = -\frac{\hbar^2}{2M} \langle \psi_{\ell} | [\Delta (\phi_{\ell} |\psi_{\ell}\rangle)] + E_{\ell} \phi_{\ell} \quad (\text{III.86})$$

It remains to make explicit the action of the Laplacian operator:

$$\Delta (\phi_{\ell} |\psi_{\ell}\rangle) = (\Delta \phi_{\ell}) |\psi_{\ell}\rangle + 2\nabla \phi_{\ell} \cdot \nabla |\psi_{\ell}\rangle + \phi_{\ell} \Delta |\psi_{\ell}\rangle, \quad (\text{III.87})$$

a quantity that we multiply by the bra $\langle\psi_\ell|$. The first term on the right-hand side simply gives $\Delta\phi_\ell$, and the second term brings out the Berry connection:

$$\mathcal{A}_\ell(\mathbf{r}) = i\hbar\langle\psi_\ell|\nabla\psi_\ell\rangle. \quad (\text{III.88})$$

Using $\hat{p} = -i\hbar\nabla$, the second term of (III.87) is written

$$2\nabla\phi_\ell \cdot \langle\psi_\ell|\nabla\psi_\ell\rangle = \frac{2}{\hbar^2} (\mathcal{A}_\ell \cdot \hat{p}) \phi_\ell. \quad (\text{III.89})$$

Finally, the equation of motion for the probability amplitude $\phi_\ell(\mathbf{r}, t)$ can be cast in the form of an effective Schrödinger equation:

$$i\hbar\frac{\partial\phi_\ell}{\partial t} = \left[\frac{(\hat{p} - \mathcal{A}_\ell(\mathbf{r}))^2}{2M} + E_\ell(\mathbf{r}) + \mathcal{V}_\ell(\mathbf{r}) \right] \phi_\ell(\mathbf{r}, t), \quad (\text{III.90})$$

where we have introduced the *scalar potential* $\mathcal{V}_\ell$, which collects all the terms depending on $\mathbf{r}$ that multiply $\phi_\ell(\mathbf{r}, t)$:

$$\begin{aligned} \mathcal{V}_\ell(\mathbf{r}) &= -i\frac{\hbar}{2M}\nabla \cdot \mathcal{A}_\ell - \frac{\mathcal{A}_\ell^2(\mathbf{r})}{2M} - \frac{\hbar^2}{2M}\langle\psi_\ell|(\Delta|\psi_\ell\rangle) \\ &= \frac{\hbar^2}{2M}(\langle\nabla\psi_\ell|\cdot|\nabla\psi_\ell\rangle - \langle\nabla\psi_\ell|\psi_\ell\rangle \cdot \langle\psi_\ell|\nabla\psi_\ell\rangle) \\ &= \frac{\hbar^2}{2M}\sum_{n\neq\ell}|\langle\nabla\psi_\ell|\psi_n\rangle|^2 \end{aligned} \quad (\text{III.91})$$

The form (III.90) for the equation of motion of $\phi_\ell(\mathbf{r}, t)$ is very suggestive. It is identical to the one governing the motion of a charged particle evolving in a magnetic field $\mathbf{B}(\mathbf{r})$ derived from the vector potential $\mathbf{A}(\mathbf{r})$. More precisely, the Berry connection $\mathcal{A}_\ell(\mathbf{r})$ plays the role of the vector potential $\mathbf{A}(\mathbf{r})$ if one assigns, by convention, a charge $q = 1$ to the particle. This confirms the analogy already anticipated between the Aharonov–Bohm phase $\Phi^{\text{A-B}}$ and the Berry geometric phase

$$\Phi^{\text{A-B}}(\mathcal{C}) = \frac{q}{\hbar} \oint_{\mathcal{C}} \mathbf{A}(\mathbf{r}) \cdot d\mathbf{r} \longleftrightarrow \Phi_\ell^{\text{Berry}}(\mathcal{C}) = \frac{1}{\hbar} \oint_{\mathcal{C}} \mathcal{A}_\ell(\mathbf{r}) \cdot d\mathbf{r} \quad (\text{III.92})$$

In addition to the artificial orbital magnetism appearing in (III.90), two terms contribute to the dynamics of the particle. The first is simply the

energy $E_\ell(\mathbf{r})$, associated with the internal state $|\psi_\ell\rangle$ of the particle. The origin of this term, which was also at the origin of the dynamical phase (III.33), is quite clear, and we will not comment further on it. The origin of the scalar potential $\mathcal{V}(\mathbf{r})$ is more subtle and deserves to be explored, which we will do in the following paragraph.

4-3 Physical interpretation of the scalar potential

To understand the origin of the scalar potential $\mathcal{V}_\ell(\mathbf{r})$ appearing in (III.90), we will return to the semiclassical formalism, which consists in treating the internal degrees of freedom of the particle quantum-mechanically, but its external degrees of freedom (position $\mathbf{r}$ and momentum $\mathbf{p}$ of the center of mass) classically.

Let us begin by defining the force operator $\hat{\mathbf{F}}(\mathbf{r})$ deduced from the initial Hamiltonian:

$$\hat{\mathbf{F}} = -\nabla_{\mathbf{r}} (\hat{H}_{\text{tot}}) = -\nabla_{\mathbf{r}} (\hat{H}_{\text{int}}(\mathbf{r})). \quad (\text{III.93})$$

We note that a particle prepared in the internal state ψ_ℓ is in an eigenstate of $\hat{H}_{\text{int}}$, but not in an eigenstate of $\hat{\mathbf{F}}$. In other words:

$$\langle\hat{\mathbf{F}}^2\rangle \neq \langle\hat{\mathbf{F}}\rangle^2, \quad (\text{III.94})$$

which means that the force acting on the atom undergoes quantum fluctuations about its average value. We will see that these fluctuations occur at the Bohr frequencies $\omega_{j\ell} = (E_j - E_\ell)/\hbar$ of the internal quantum motion. The adiabatic approximation takes advantage of the fact that these frequencies are much higher than the characteristic frequencies of the center-of-mass motion. As a result, the quantum fluctuations of the force create a micro-motion of the particle, similar to the one that occurs for charged particles in a Paul trap. The kinetic energy of this micro-motion plays the role of a potential energy for the center-of-mass motion, which is the origin of the scalar potential $\mathcal{V}(\mathbf{r})$ (Aharonov & Stern 1992; Cheneau, Rath, et al. 2008).

Force operator. Starting from the following expression for $\hat{H}_{\text{int}}(\mathbf{r})$

$$\hat{H}_{\text{int}}(\mathbf{r}) = \sum_n E_n(\mathbf{r}) \hat{P}_n(\mathbf{r}) \quad \text{with} \quad \hat{P}_n(\mathbf{r}) = |\psi_n(\mathbf{r})\rangle\langle\psi_n(\mathbf{r})|, \quad (\text{III.95})$$

we obtain the expression for the force operator:

$$\hat{\mathbf{F}}(\mathbf{r}) = \sum_n (\nabla E_n) \hat{P}_n + E_n (\nabla \hat{P}_n) \quad (\text{III.96})$$

with

$$\nabla \hat{P}_n = |\nabla \psi_n\rangle \langle \psi_n| + |\psi_n\rangle \langle \nabla \psi_n|. \quad (\text{III.97})$$

Average force. It is simple to compute the average value of $\hat{\mathbf{F}}$ when the particle is in the state ψ_ℓ . The fact that

$$\langle \psi_\ell | \psi_\ell \rangle = 1 \Rightarrow \langle \nabla \psi_\ell | \psi_\ell \rangle + \langle \psi_\ell | \nabla \psi_\ell \rangle = 0 \quad (\text{III.98})$$

implies that the contribution of the term in $\nabla \hat{P}_n$ in (III.96) vanishes, and we simply obtain

$$\langle \psi_\ell | \hat{\mathbf{F}} | \psi_\ell \rangle = -\nabla E_\ell. \quad (\text{III.99})$$

This result is easy to interpret: since the particle is on an internal level of energy $E_\ell(\mathbf{r})$, its center of mass experiences the force field derived from this energy. This contribution is indeed recovered as a potential energy in (III.90).

Fluctuations of the force. The fluctuations of the force operator when the particle is prepared in the internal state ψ_ℓ are characterized by the correlation function

$$C(t, t') = \frac{1}{2} \langle \delta \hat{\mathbf{F}}(\mathbf{r}, t) \cdot \delta \hat{\mathbf{F}}(\mathbf{r}, t') + \delta \hat{\mathbf{F}}(\mathbf{r}, t') \cdot \delta \hat{\mathbf{F}}(\mathbf{r}, t) \rangle, \quad (\text{III.100})$$

where we have set $\delta \hat{\mathbf{F}} = \hat{\mathbf{F}} - \langle \hat{\mathbf{F}} \rangle$, and where the operators in (III.100) are computed in the Heisenberg picture. Let us recall this picture; operators such as

$$\hat{Q}_{n,\ell}(\mathbf{r}) = |\psi_n(\mathbf{r})\rangle \langle \psi_\ell(\mathbf{r})| \quad (\text{III.101})$$

depend on time according to the equation

$$i\hbar \dot{\hat{Q}} = [\hat{Q}, \hat{H}] \quad (\text{III.102})$$

which gives

$$\hat{Q}_{n,\ell}(\mathbf{r}, t) = \hat{Q}_{n,\ell}(\mathbf{r}, 0) e^{i\omega_{n\ell}t}. \quad (\text{III.103})$$

A relatively long, but straightforward, calculation then leads to:

$$C(\tau) = \sum_{n \neq \ell} C_n \cos(\omega_{n\ell}\tau), \quad (\text{III.104})$$

where we have set $\tau = t - t'$ and

$$C_n = \hbar^2 \omega_{n\ell}^2 |\langle \psi_\ell | \nabla \psi_n \rangle|^2. \quad (\text{III.105})$$

Micro-motion. To analyze the consequences of the quantum fluctuations of the force, let us consider the equivalent classical problem; suppose that a classical particle is subject to a stochastic force $\mathbf{F}(t)$ such that:

$$\overline{\mathbf{F}(t)} = 0, \quad \overline{\mathbf{F}(t) \cdot \mathbf{F}(t')} = C(t - t'). \quad (\text{III.106})$$

The Fourier transform $\mathbf{f}(\omega)$ of $\mathbf{F}(t)$ is such that

$$\overline{\mathbf{f}(\omega)} = 0, \quad \overline{\mathbf{f}^*(\omega) \cdot \mathbf{f}(\omega')} = \delta(\omega - \omega') B(\omega) \quad (\text{III.107})$$

where

$$B(\omega) = \frac{1}{2} \sum_{n \neq \ell} C_n [\delta(\omega + \omega_{n\ell}) + \delta(\omega - \omega_{n\ell})] \quad (\text{III.108})$$

is the Fourier transform of $C(\tau)$. The solution of the equation of motion $\dot{\mathbf{p}} = \mathbf{F}$ is

$$\mathbf{p}(t) = \int \frac{\mathbf{f}(\omega)}{i\omega} e^{i\omega t} d\omega \quad (\text{III.109})$$

which corresponds to a zero average momentum. On the other hand, the associated average kinetic energy is strictly positive and equal to

$$\frac{\overline{\mathbf{p}^2}}{2M} = \int \frac{B(\omega)}{2M\omega^2} d\omega = \frac{1}{2M} \sum_{n \neq \ell} \frac{C_n}{\omega_{n\ell}^2} \quad (\text{III.110})$$

$$= \frac{\hbar^2}{2M} \sum_{n \neq \ell} |\langle \psi_\ell | \nabla \psi_n \rangle|^2 = \mathcal{V}_\ell(\mathbf{r}). \quad (\text{III.111})$$

In the explicit calculation of (III.111), the contribution of the ω^{-2} denominator cancels with the $\omega_{n\ell}$ terms that appear in expression (III.105) for C_n . In the end, we recover exactly the result (III.91) for the scalar potential, which

justifies interpreting this potential as a kinetic energy associated with the micro-motion.

In conclusion, both the vector potential $\mathcal{A}_\ell$ and the scalar potential $\mathcal{V}_\ell$ can be understood as signatures of the existence of an internal Hilbert space associated with the particle. When one makes the adiabatic approximation and assumes that the particle follows the level ψ_ℓ , one might naively believe that the only thing that matters is the energy profile $E_\ell(\mathbf{r})$. This is not the case: the existence of the other internal states ψ_n , even if they are not populated, manifests itself through these two potentials, giving rise to a nonzero quantum phase over a closed trajectory related to the flux of the Berry curvature $\mathcal{B}_\ell$ (III.47), and modifying the energy profile via the scalar potential $\mathcal{V}_\ell$ (III.91).

Chapter IV

Artificial magnetism for an isolated atom

At the end of the previous lectures, we now have several avenues for simulating and characterizing the orbital magnetism of an assembly of neutral particles. We wish to arrive at a Hamiltonian of the type:

$$\hat{H} = \frac{(\hat{\mathbf{p}} - \mathbf{A}(\hat{\mathbf{r}}))^2}{2M} + \dots = \frac{\hat{\mathbf{p}}^2}{2M} - \frac{\hat{\mathbf{p}} \cdot \mathbf{A}(\hat{\mathbf{r}})}{M} + \dots \quad (\text{IV.1})$$

where the vector potential is “nontrivial,” i.e.,

$$\mathbf{B} = \nabla \times \mathbf{A} \neq 0. \quad (\text{IV.2})$$

In particular, the practically very important case of a uniform magnetic field can be realized starting from vector potentials written in the symmetric gauge or the Landau gauge:

$$\mathbf{A}^{(\text{sym})}(\mathbf{r}) = \frac{B}{2} (-y \mathbf{u}_x + x \mathbf{u}_y), \quad \mathbf{A}^{(\text{Landau})}(\mathbf{r}) = B x \mathbf{u}_y. \quad (\text{IV.3})$$

To structure the presentation of the main approaches that have recently been explored in this research field, one can begin by identifying several questions:

- *What confinement for the atoms?* We will consider two possible answers to this question. One can first be interested in free atoms or atoms confined in a harmonic trap of fairly weak stiffness, for which one recovers almost exactly the notion of Landau levels already encountered.

Conversely, one can consider one or more particles confined in an optical lattice in the tight-binding approximation; this “discretization of space” somewhat changes the nature of the problem, which then reduces to the study of the topological properties of the relevant energy bands.

- *Do we wish to take advantage of the internal structure of the atoms?* Unlike electrons in solids, our atoms possess a series of internal states that can be coupled to one another by means of light beams. This degree of freedom makes it possible, in particular, to envisage the notion of adiabatically following a dressed level, together with the Berry phase and the geometric gauge field that comes with it.
- *Does the Hamiltonian depend explicitly on time?* This possibility of temporally modulating the potential acting on the atoms is also a very useful degree of freedom. This potential can have a frequency of the same order as the cyclotron frequency ω_c that one seeks to make appear, which is what we will encounter with the rotation of a system. This modulation frequency can also be much larger than ω_c , the gauge field then appearing as a secular term resulting from a well-chosen micro-motion. We will see examples of this for atoms confined in an optical lattice.

There is (to our knowledge) no universal criterion for favoring one approach over another. The use of several internal states is, for example, a tool offering great flexibility, but it raises the question of possible inelastic

collisions between atoms, which are not a problem if the atoms are all confined to the internal state of lowest energy. We will therefore compare, as we progress, the advantages and drawbacks associated with each of these methods. In this chapter, we will thus concentrate on atoms that are free or confined in an isotropic harmonic trap of angular frequency ω . We will review two methods leading to the model Hamiltonian (IV.1), the first based on setting the system into rotation and the second on the explicit use of several atomic internal states. One may consult the recent review article by Goldman, Juzeliunas, et al. (2014) for a deeper look at the notions developed in this chapter [see also the article by Dalibard, Gerbier, et al. (2011) for a more concise review].

1 Physics in a rotating frame

Starting from a fluid composed of neutral particles of mass M , one of the conceptually simplest methods for simulating orbital magnetism consists in “setting it in rotation.” One considers an assembly of neutral particles, and arranges for their environment (the trap confining them) to rotate about the z axis with angular velocity Ω : the Hamiltonian describing the external forces acting on the atoms is thus modulated in time with this angular frequency Ω . One can then show that the dynamics of the particles of the fluid is equivalent to that of particles of charge q placed in a magnetic field such that

$$q\mathbf{B} = 2M\mathbf{\Omega}, \quad \text{with} \quad \mathbf{\Omega} = \Omega \mathbf{u}_z. \quad (\text{IV.4})$$

This similarity between orbital magnetism and rotation is easily understood by comparing the Lorentz force responsible for magnetism and the Coriolis force that appears in a rotating frame:

$$\mathbf{F}_{\text{Lorentz}} = q \mathbf{v} \times \mathbf{B}, \quad \mathbf{F}_{\text{Coriolis}} = 2M \mathbf{v} \times \mathbf{\Omega}. \quad (\text{IV.5})$$

The goal of this paragraph is to formalize this similarity within the Hamiltonian formalism of quantum physics, and to bring out the physical meaning of the relevant observables.

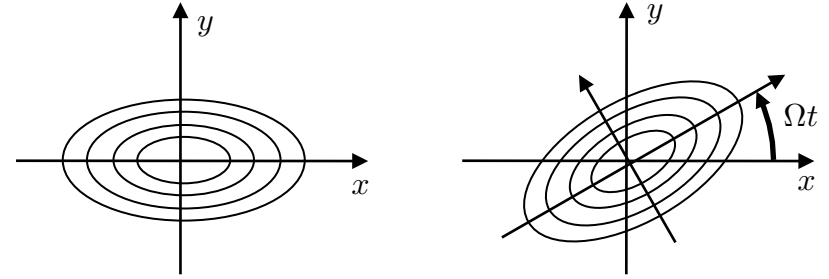


Figure IV.1. Left: equipotential curves for $V(\mathbf{r})$. Right: equipotential curves at a given instant for $W(\mathbf{r}, t)$, corresponding to a rotation at angular velocity Ω of the initial potential $V(\mathbf{r})$.

1-1 Setting a quantum system into rotation

Let us begin by considering a single particle of mass M , subject, in a Galilean frame $\mathcal{R}$, to the time-independent potential $V(\mathbf{r})$. The quantum motion of the particle is described by the Schrödinger equation with a time-independent Hamiltonian:

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi \quad \text{with} \quad \hat{H} = \frac{\hat{\mathbf{p}}^2}{2M} + V(\hat{\mathbf{r}}). \quad (\text{IV.6})$$

The potential $V(\hat{\mathbf{r}})$ describes the set of external factors acting on the particle: a confining potential in the case of an experiment with a trapped gas, mirrors and beam splitters in the case of an interferometry experiment. In particular, for the study of a gas of particles, and as long as the Hamiltonian is time-independent, the usual procedure of quantum statistical physics applies: finding the eigenstates, writing a thermal equilibrium state, computing the various equilibrium quantities, etc.

Suppose now that we rotate the potential about the z axis at constant angular velocity Ω . The single-particle Hamiltonian in the frame $\mathcal{R}$ then depends explicitly on time:

$$\hat{H}(t) = \frac{\hat{\mathbf{p}}^2}{2M} + W(\hat{\mathbf{r}}, t) \quad (\text{IV.7})$$

with (cf. figure IV.1):

$$W(x, y, z, t) = V[x \cos(\Omega t) + y \sin(\Omega t), -x \sin(\Omega t) + y \cos(\Omega t), z]. \quad (\text{IV.8})$$

As soon as the Hamiltonian depends explicitly on time, it is no longer possible to apply the usual procedure to construct a thermal equilibrium state. Nevertheless, in this case one can reduce the problem to a time-independent one via a unitary transformation: the usual methods of quantum thermodynamics then become usable again.

Remark. We assume here that the potential $V(\mathbf{r})$ is not invariant under rotation about the z axis. Otherwise, the potential $W(\mathbf{r}, t)$ does not actually depend on time, and the frame $\mathcal{R}$ is a legitimate frame in which to apply the concepts of statistical physics.

1-2 Unitary transformation

Let us first recall the general principles of a unitary transformation¹. Let us start from a system described by a time-dependent Hamiltonian $H(t)$, with a state evolving according to the Schrödinger equation:

$$i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H}(t) |\psi(t)\rangle. \quad (\text{IV.9})$$

We introduce an operator $\hat{U}(t)$ and construct the transformed states

$$|\tilde{\psi}(t)\rangle = \hat{U}(t) |\psi(t)\rangle. \quad (\text{IV.10})$$

The evolution equation for $|\tilde{\psi}(t)\rangle$ is also given by a Schrödinger equation, with the Hamiltonian $\hat{\tilde{H}}$:

$$i\hbar \frac{\partial |\tilde{\psi}(t)\rangle}{\partial t} = \hat{\tilde{H}} |\tilde{\psi}(t)\rangle, \quad \hat{\tilde{H}} = \hat{U} \hat{H} \hat{U}^\dagger + i\hbar \frac{d\hat{U}}{dt} \hat{U}^\dagger. \quad (\text{IV.11})$$

¹Throughout this paragraph we adopt a quantum point of view, well suited to the applications we will consider afterward. Note, however, that a classical treatment based on the Lagrangian formalism gives formally identical results. For such a treatment, see for example the book by Landau & Lifshitz (1982), § 39, as well as C. Cohen-Tannoudji's 2001-02 lecture course at the Collège de France (slides 80-85).

Our goal in what follows is to propose an operator $\hat{U}$ such that the transform $\hat{\tilde{H}}$ of the Hamiltonian given in (IV.7-IV.8) is time-independent.

Let us set

$$\hat{U}(t) = \exp(i\Omega t \hat{L}_z / \hbar) \quad \text{with} \quad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = -i\hbar \frac{\partial}{\partial \varphi}, \quad (\text{IV.12})$$

where the last expression refers to cylindrical coordinates (r, φ, z) with $x = r \cos \varphi$, $y = r \sin \varphi$. The action of $\hat{U}$ on a function $\psi(r, \varphi, z)$ is

$$\begin{aligned} \hat{U}\psi(r, \varphi, z) &= \exp\left(\Omega t \frac{\partial}{\partial \varphi}\right) \psi(r, \varphi, z) \\ &= \psi(r, \varphi, z) + \Omega t \frac{\partial \psi}{\partial \varphi} + \frac{1}{2}(\Omega t)^2 \frac{\partial^2 \psi}{\partial \varphi^2} + \dots \\ &= \psi(r, \varphi + \Omega t, z). \end{aligned} \quad (\text{IV.13})$$

Let us examine how the various operators appearing in (IV.7) transform under the action of $\hat{U}$. Let us begin with $\hat{x}$ and $\hat{y}$, for which one finds:

$$\hat{U}\hat{x}\hat{U}^\dagger = \hat{x} \cos(\Omega t) - \hat{y} \sin(\Omega t), \quad \hat{U}\hat{y}\hat{U}^\dagger = \hat{x} \sin(\Omega t) + \hat{y} \cos(\Omega t). \quad (\text{IV.14})$$

To demonstrate these identities, let us consider, for example, the action of $\hat{U}\hat{x}\hat{U}^\dagger$ on a function $\psi(r, \varphi, z)$:

$$\begin{aligned} (\hat{U}\hat{x}\hat{U}^\dagger) \psi(r, \varphi, z) &= (\hat{U}\hat{x}) \psi(r, \varphi - \Omega t, z) \\ &= \hat{U} [r \cos \varphi \psi(r, \varphi - \Omega t, z)] \\ &= r \cos(\varphi + \Omega t) \psi(r, \varphi, z) \\ &= (x \cos(\Omega t) - y \sin(\Omega t)) \psi(r, \varphi, z). \end{aligned} \quad (\text{IV.15})$$

The result (IV.14) can be inverted to give

$$\hat{U} (\hat{x} \cos(\Omega t) + \hat{y} \sin(\Omega t)) \hat{U}^\dagger = \hat{x}, \quad \hat{U} (-\hat{x} \sin(\Omega t) + \hat{y} \cos(\Omega t)) \hat{U}^\dagger = \hat{y}, \quad (\text{IV.16})$$

and thus

$$\begin{aligned} \hat{U}W(\hat{\mathbf{r}}, t)\hat{U}^\dagger &= \hat{U}V[\hat{x} \cos(\Omega t) + \hat{y} \sin(\Omega t)t, -\hat{x} \sin(\Omega t) + \hat{y} \cos(\Omega t), \hat{z}] \hat{U}^\dagger \\ &= V(\hat{x}, \hat{y}, \hat{z}) \end{aligned} \quad (\text{IV.17})$$

which is time-independent.

Let us now consider the kinetic-energy term $\hat{p}^2/2m$ of (IV.7). Using the Laplacian operator in cylindrical coordinates:

$$\frac{\hat{p}^2}{2M} = -\frac{\hbar^2}{2M}\Delta = -\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right), \quad (\text{IV.18})$$

one sees that this operator commutes with $\hat{L}_z = -i\hbar\partial_\varphi$, which makes sense since this kinetic-energy term is invariant under rotation.

The last term to compute in order to obtain the expression for $\hat{H}$ is

$$i\hbar\frac{d\hat{U}}{dt}\hat{U}^\dagger = i\hbar\left(\frac{i\Omega\hat{L}_z}{\hbar}\hat{U}\right)\hat{U}^\dagger = -\Omega\hat{L}_z, \quad (\text{IV.19})$$

which is also time-independent.

In the end, we obtain for $\hat{H}$ the following expression

$$\hat{H} = \frac{\hat{p}^2}{2M} + V(\hat{\mathbf{r}}) - \Omega\hat{L}_z, \quad (\text{IV.20})$$

which is indeed time-independent: we will be able to use this Hamiltonian as the basis for a study of equilibrium thermodynamics.

1-3 Physical interpretation

The unitary transformation described in the previous paragraph corresponds to the well-known change of frame, going from the Galilean frame $\mathcal{R}$ to the frame rotating at angular velocity Ω about the z axis, denoted $\mathcal{R}'$.

To show this correspondence, let us start from a basis $\mathbf{u}_j$ ($j = x, y, z$) attached to the frame $\mathcal{R}$. Let us introduce the basis $\mathbf{u}'_j$ attached to $\mathcal{R}'$ (cf. figure IV.2):

$$\mathbf{u}'_x = \cos(\Omega t)\mathbf{u}_x + \sin(\Omega t)\mathbf{u}_y, \quad \mathbf{u}'_y = -\sin(\Omega t)\mathbf{u}_x + \cos(\Omega t)\mathbf{u}_y, \quad \mathbf{u}'_z = \mathbf{u}_z. \quad (\text{IV.21})$$

A point M with coordinates (x, y, z) in the basis $\mathbf{u}_j$ has coordinates:

$$x' = \cos(\Omega t)x + \sin(\Omega t)y, \quad y' = -\sin(\Omega t)x + \cos(\Omega t)y, \quad z' = z \quad (\text{IV.22})$$

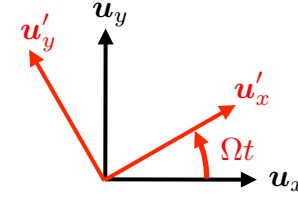


Figure IV.2. Fixed basis $(\mathbf{u}_x, \mathbf{u}_y)$ of the Galilean frame $\mathcal{R}$ and rotating basis $\mathbf{u}'_x, \mathbf{u}'_y$ attached to the rotating frame $\mathcal{R}'$.

in the basis $\mathbf{u}'_j$, which ensures that

$$x\mathbf{u}_x + y\mathbf{u}_y + z\mathbf{u}_z = x'\mathbf{u}'_x + y'\mathbf{u}'_y + z'\mathbf{u}'_z. \quad (\text{IV.23})$$

Comparing (IV.22) with the result obtained in (IV.16)

$$\hat{U}(\hat{x}\cos(\Omega t) + \hat{y}\sin(\Omega t))\hat{U}^\dagger = \hat{x}$$

then leads to the following interpretation: the operator $\hat{x}$ in the “new representation,” i.e., the one obtained after the unitary transformation, corresponds to the operator $\hat{x}\cos(\Omega t) + \hat{y}\sin(\Omega t)$ in the old representation. Now, according to (IV.22), this operator is nothing other than the position operator associated with the coordinate x' in the basis $\mathbf{u}'_j$ of the rotating frame.

The action of $\hat{U}$ thus indeed corresponds to a change of frame, the new position operators corresponding to the physical quantity “position in the basis $\mathbf{u}'_j$ attached to the frame $\mathcal{R}'$.”

The presence of the term $-\Omega\hat{L}_z$, as for the magnetic Hamiltonian, modifies the usual relation between velocity and momentum. One can define the velocity operator from the Hamiltonian (IV.20) by working in the Heisenberg picture:

$$\hat{\mathbf{v}} = \frac{d\hat{\mathbf{r}}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{\mathbf{r}}] \quad (\text{IV.24})$$

which leads to

$$\hat{v}_x = \frac{\hat{p}_x}{M} + \Omega\hat{y}, \quad \hat{v}_y = \frac{\hat{p}_y}{M} - \Omega\hat{x}, \quad \hat{v}_z = \frac{\hat{p}_z}{M}. \quad (\text{IV.25})$$

In other words, in the new representation, the momentum $\hat{\mathbf{p}}$ and the kinetic momentum $M\hat{\mathbf{v}}$ are related by

$$\hat{\mathbf{p}} = M\hat{\mathbf{v}} + M\boldsymbol{\Omega} \times \hat{\mathbf{r}}. \quad (\text{IV.26})$$

To interpret this result, let us return for a moment to classical kinematics. A particle with coordinates (x, y, z) in the basis $\mathbf{u}_j$ attached to $\mathcal{R}$ has, in this frame, velocity

$$\mathbf{v}_{\mathcal{R}} = \dot{x}\mathbf{u}_x + \dot{y}\mathbf{u}_y + \dot{z}\mathbf{u}_z. \quad (\text{IV.27})$$

In the basis $\mathbf{u}'_j$ attached to the frame $\mathcal{R}'$, the coordinates of the particle are (x', y', z') and its velocity is:

$$\mathbf{v}_{\mathcal{R}'} = \dot{x}'\mathbf{u}'_x + \dot{y}'\mathbf{u}'_y + \dot{z}'\mathbf{u}'_z. \quad (\text{IV.28})$$

The relation between $\mathbf{v}_{\mathcal{R}}$ and $\mathbf{v}_{\mathcal{R}'}$ is obtained by taking into account the rotation of the $\mathbf{u}'_j$ relative to the $\mathbf{u}_j$:

$$\frac{d\mathbf{u}'_j}{dt} = \boldsymbol{\Omega} \times \mathbf{u}'_j \quad (\text{IV.29})$$

from which we deduce

$$x\mathbf{u}_x + y\mathbf{u}_y + z\mathbf{u}_z = x'\mathbf{u}'_x + y'\mathbf{u}'_y + z'\mathbf{u}'_z \Rightarrow \mathbf{v}_{\mathcal{R}} = \mathbf{v}_{\mathcal{R}'} + \boldsymbol{\Omega} \times \mathbf{r}'. \quad (\text{IV.30})$$

This law of composition of velocities makes it possible to interpret relation (IV.26). In (IV.26), the operator $\hat{\mathbf{v}}$ is associated with the physical quantity *velocity in the rotating frame*, since it is the derivative of the operator *position in the rotating frame*. We deduce from (IV.30) that the momentum operator $\hat{\mathbf{p}}$ in the new representation is associated with the physical quantity *mass $\times$ velocity in the Galilean frame $\mathcal{R}$* .

To conclude, let us examine the operator “ z component of the angular momentum.” It is clear that $\hat{L}_z$ commutes with $\hat{U}$, i.e.,

$$\hat{U}\hat{L}_z\hat{U}^\dagger = \hat{L}_z. \quad (\text{IV.31})$$

The operator $\hat{L}_z$ thus always corresponds to the angular momentum in the Galilean frame $\mathcal{R}$.

1-4 Connection between orbital magnetism and rotation

We now have all the elements needed to show the equivalence between orbital magnetism and rotation. Consider the Hamiltonian of a charged particle in a uniform magnetic field $\mathbf{B} = B\mathbf{u}_z$, and let us choose the symmetric gauge:

$$\mathbf{A}(\mathbf{r}) = \frac{B}{2}(-y\mathbf{u}_x + x\mathbf{u}_y) \quad (\text{IV.32})$$

which gives (cf. chapter 2):

$$\hat{H} = \frac{(\hat{\mathbf{p}} - q\mathbf{A}(\hat{\mathbf{r}}))^2}{2M} = \frac{\hat{p}^2}{2M} + \frac{1}{8}M\omega_c^2 r^2 - \frac{\omega_c}{2}\hat{L}_z. \quad (\text{IV.33})$$

with $\omega_c = qB/M$. The Hamiltonian in the rotating frame (IV.20) has a structure identical to this magnetic Hamiltonian, provided one takes

$$\Omega = \omega_c/2 \quad (\text{IV.34})$$

and chooses the external potential $V(r)$ such that

$$V(r) = \frac{1}{8}M\omega_c^2 r^2 = \frac{1}{2}M\Omega^2 r^2. \quad (\text{IV.35})$$

This choice means that the restoring force associated with $V(\mathbf{r})$ exactly compensates the centrifugal force.

Even if the potential $V(\mathbf{r})$ is not taken exactly opposite to the centrifugal potential, one can still write the Hamiltonian in the rotating frame (IV.20) as

$$\hat{H} = \frac{(\hat{\mathbf{p}} - q\mathbf{A}(\hat{\mathbf{r}}))^2}{2M} + V(\mathbf{r}) + V_{\text{centrif.}}(r) \quad (\text{IV.36})$$

with

$$q\mathbf{A} = M\Omega(-y\mathbf{u}_x + x\mathbf{u}_y), \quad V_{\text{centrif.}}(r) = -\frac{1}{2}M\Omega^2 r^2 \quad (\text{IV.37})$$

which means that one thereby simulates the motion of a charged particle placed in a uniform magnetic field associated with the cyclotron frequency $\omega_c = 2\Omega$, and subject in addition to the total potential $V + V_{\text{centrif.}}$.

2 A rotating gas

2-1 Rotating anisotropy

To set a gas of cold atoms into rotation, a simple method consists in starting from an isotropic trap in the xy plane, corresponding to the static potential

$$V_0(r) = \frac{1}{2}M\omega^2(x^2 + y^2) \quad (\text{IV.38})$$

and adding a slight anisotropy rotating at the angular frequency Ω :

$$\begin{aligned} \delta V(x, y, t) &= \frac{\epsilon}{2}M\omega^2 [y'(t)^2 - x'(t)^2] \\ &= \frac{\epsilon}{2}M\omega^2 [(-\sin(\Omega t)x + \cos(\Omega t)y)^2 - (\cos(\Omega t)x + \sin(\Omega t)y)^2] \\ &= \frac{\epsilon}{2}M\omega^2 [\cos(2\Omega t)(y^2 - x^2) - 2\sin(2\Omega t)xy] \end{aligned} \quad (\text{IV.39})$$

where the parameter ϵ characterizes the amplitude of the anisotropy. In the rotating frame, one obtains a slightly anisotropic static potential:

$$V(x, y) = V_0(x, y) + \delta V(x, y) = \frac{1}{2}M\omega^2 ((1 - \epsilon)x^2 + (1 + \epsilon)y^2) \quad (\text{IV.40})$$

In practice, this rotating potential can be created by an elliptical laser beam or by a time-dependent magnetic-field gradient. We show in figure IV.3 a result obtained with a rubidium gas set into rotation by a laser stirrer. The presence of a regular lattice of vortices is the unmistakable signature of this rotation, as we will have the opportunity to explain in detail in a forthcoming chapter.

2-2 Can the critical rotation $\Omega = \omega$ be reached?

We explained at the end of the previous paragraph that the ideal case of a free particle placed in a uniform magnetic field is reached when the confining potential $V(r)$ exactly compensates the centrifugal potential. It is in this case, and only in this case, that the single-particle spectrum of the Hamiltonian consists of degenerate Landau levels, this degeneracy being

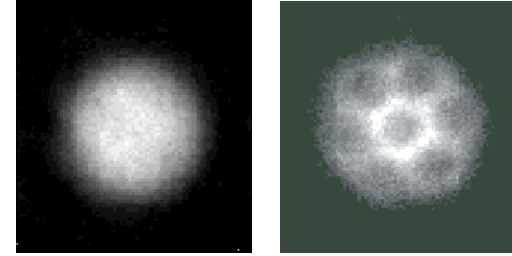


Figure IV.3. Left: Bose-Einstein condensate of rubidium atoms confined in an isotropic trap in the xy plane. Right: lattice of vortices obtained by adding a rotating anisotropy (Eqs. IV.38-IV.39) with $\epsilon \approx 0.1$. Figure taken from Madison, Chevy, et al. (2000b).

an important ingredient for obtaining strongly correlated states in the presence of interactions.

The presence of the anisotropy ϵ will prevent this regime from being reached. More precisely, one can show that the rotation range

$$\Omega/\omega \in [1 - \epsilon, 1 + \epsilon] \quad (\text{IV.41})$$

corresponds to a dynamical instability for the motion of the particles (Rosenbusch, Petrov, et al. 2002). This result is obtained fairly simply in Newtonian dynamics by writing the equations of motion in the rotating frame, taking into account the Coriolis force and the centrifugal force, as well as the force associated with the potential $V_0 + \delta V$:

$$\begin{aligned} \ddot{x} - 2\Omega\dot{y} + [\omega^2(1 - \epsilon) - \Omega^2]x &= 0, \\ \ddot{y} + 2\Omega\dot{x} + [\omega^2(1 + \epsilon) - \Omega^2]y &= 0. \end{aligned}$$

Finding the eigenvalues of this system leads to the announced result. For example, if one chooses exactly $\Omega = \omega$, one finds that the quantity $x + y$ diverges as $\exp(\epsilon\omega t/2)$. An example of this divergence is shown in figure IV.4.

This instability thus prevents the angular rotation frequency Ω from being taken as close as one would like to the trap angular frequency ω : one is limited to $\Omega \lesssim \omega(1 - \epsilon)$. This would not be a serious issue if ϵ itself could

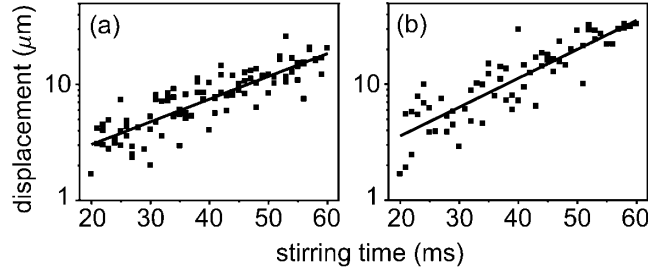


Figure IV.4. Dynamical instability of a cold-atom cloud (non-condensed on the left, nearly pure condensate on the right) for a rotation frequency equal to the trapping frequency $\Omega = \omega$. An exponential growth of the offset between the center of the trap and the center of gravity of the cloud is observed. The anisotropy parameter ϵ is equal to 0.09 for this experiment carried out with ^{87}Rb rubidium atoms ($\omega/(2\pi) = 180 \text{ Hz}$). Figure taken from the article by Rosenbusch, Petrov, et al. (2002).

be made arbitrarily small. But a real trap is never perfectly isotropic, and is thus characterized by a nonzero static anisotropy $\epsilon_{\text{stat.}}$. For the transition to the rotating frame to be relevant, the rotating anisotropy ϵ must be large compared to this static anisotropy, which in practice forces one to take parameters ϵ of a few percent. We will see in a forthcoming lecture that this constraint leaves us relatively far from the physics of the quantum Hall effect.

2-3 Rotationally invariant systems

There is an interesting alternative to going into the rotating frame in order to generate the term in $-\Omega \hat{L}_z$, later reinterpreted as a term in $-\hat{\mathbf{p}} \cdot \mathbf{A}(\hat{\mathbf{r}})$. This alternative consists in injecting a nonzero average total angular momentum $\langle L_z \rangle$ into a gas confined in a trap that is approximately rotationally invariant. The gas is then left to reach a stationary thermodynamic state, given that this angular momentum is a conserved quantity.

The standard approach for finding this stationary state is the technique of Lagrange multipliers, described for example in the book by Diu, Guth-

mann, et al. (1989), complement II.E (one may also consult the book by Landau & Lifchitz (1984), § 26 and § 34). Suppose that we are interested in the state at zero temperature, i.e., the ground state given the constraint on the angular momentum. A Lagrange parameter Ω is introduced to account for this constraint, and one seeks the state Ψ that minimizes the average energy

$$E(\Omega, \Psi) = \langle \Psi | \hat{H}_\Omega | \Psi \rangle \quad \text{with} \quad \hat{H}_\Omega = \hat{H} - \Omega \hat{L}_z. \quad (\text{IV.42})$$

The value of Ω is then determined so as to ensure that the average of $\hat{L}_z$ is indeed the one that was injected. One is thus led to work with the same Hamiltonian as the one obtained by going into the rotating frame, even though the physical interpretation of Ω is different: in one case, it is the angular rotation frequency imposed from the outside; in the other, it is an intrinsic parameter of the system that adjusts to account for the angular momentum initially injected.

Remark. We have assumed here that the Hamiltonian $\hat{H}$ is rotationally invariant and commutes with $\hat{L}_z$. If the system considered is small enough for an exact diagonalization of $\hat{H}$ to be possible, going through $\hat{H}_\Omega$ is not necessary: one can simply look for a joint eigenbasis of $\hat{H}$ and $\hat{L}_z$, and then consider only the eigenstates that lie in the sector corresponding to the injected angular momentum. On the other hand, if one must resort to mean-field-type approximations, it is indeed the functional $E(\Omega, \Psi)$ that one must work with.

How can angular momentum be injected? Once this alternative approach to forced rotation has been identified, this question arises naturally. Among the methods that have been proposed, we will present two of them here.

- The first is the selective-evaporation method, set up by Eric Cornell and his group (Schweikhard, Coddington, et al. 2004). The idea is to first set the cloud into rotation by the “classical” method, using a rotating stirrer. The stirrer is then switched off, and the cloud finds itself confined in a trap with very good rotational symmetry about the z axis: the Boulder group estimates that the residual anisotropy of their

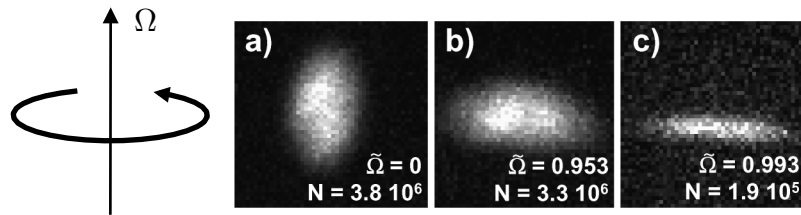


Figure IV.5. Setting a cloud into rotation by selective evaporation. The photographs are taken along an axis perpendicular to the rotation axis z . By removing atoms with below-average angular momentum, the rotation speed of the cloud remaining in the trap is increased. Panel a corresponds to a gas at rest, panel b to a gas after radiofrequency evaporation, panel c after a combination of radiofrequency and optical evaporation (a light beam propagating along the z axis ejects atoms located near this axis). The quantity $\tilde{\Omega}$ denotes Ω/ω . For the very fast rotation frequency obtained in (c), the centrifugal force considerably deforms the cloud, which is then completely flattened in the xy plane. This figure is taken from the article by Schweikhard, Coddington, et al. (2004).

trap is below 0.1%. A selective evaporation is then carried out, removing atoms located near the z axis, which therefore carry less angular momentum than average. This evaporation has the effect of increasing the average angular momentum per atom remaining in the trap (figure IV.5). The Lagrange parameter Ω , which here represents the effective rotation frequency, can reach 0.993ω in these experiments, which is a record for a purely harmonic trap. The macroscopic wave function significantly populates angular-momentum states going up to $120\hbar$.

- The second method consists in transferring angular momentum from light beams. To do this, a Raman transition induced by Laguerre–Gauss-type light beams is used. In the experiment of Moulder, Beattie, et al. (2012), the atoms are initially confined in an annular trap in an internal state $|a\rangle$ (specifically, the Zeeman state $|F = 1, m = 1\rangle$ of ^{87}Rb). An *absorption – stimulated emission* transition takes the atoms from state $|a\rangle$ to another internal state $|b\rangle$ (here $|F = 1, m = 0\rangle$). One of the beams carries an angular momentum of up to $10\hbar$, while the other is a Gaussian beam with zero orbital angular momentum. During the

Raman transition, each atom thus gains an angular momentum also going up to $10\hbar$. This creates a permanent current circulating in a ring (figure IV.6).

3 The coupling of an atom to radiation

We saw in the chapter devoted to the Berry phase how the use of an internal structure of the particles, together with the adiabatic following of a space-dependent internal state, could lead to a nonzero geometric phase and gauge field. We already indicated that the case of an atom illuminated by a coherent light beam makes it possible, in principle, to implement this strategy, by exploiting the existence of several atomic internal states coupled to one another by the radiation and the emergence of position-dependent dressed states.

To fully understand the possibilities opened up by this approach as well as its limitations, we will begin by reviewing the main elements that make it possible to describe the atom–light coupling and the way it can be implemented for different atomic species. We will then examine, in this chapter and in the one that follows, the practical implementation of this adiabatic following.

3-1 The model of a two-level atom

In what follows we consider an atom illuminated by a monochromatic light beam of angular frequency ω . We model the atom as a two-level system, denoted $|g\rangle$ and $|e\rangle$, separated by the energy $\hbar\omega_0$ (figure V.2), whose Hamiltonian in the absence of light is written:

$$\hat{H}_0 = \frac{\hbar\omega_0}{2} (|e\rangle\langle e| - |g\rangle\langle g|). \quad (\text{IV.43})$$

We assume that the frequency of the light beam is chosen relatively close to the atomic resonance:

$$|\Delta| \ll \omega_0, \omega \quad \text{with} \quad \Delta = \omega - \omega_0. \quad (\text{IV.44})$$

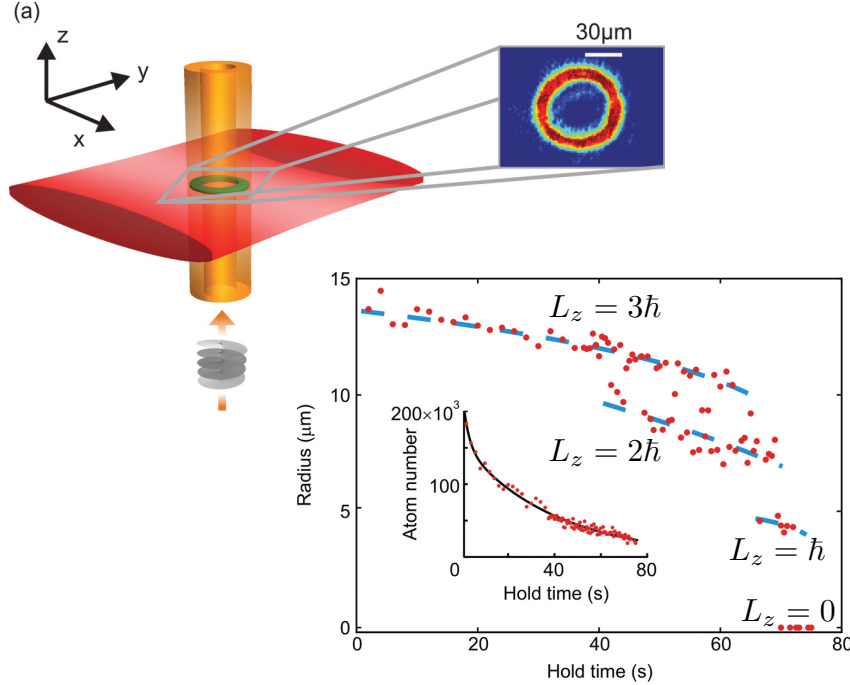


Figure IV.6. Permanent current in an annular trap, created from laser beams in a Laguerre–Gauss mode. A determined angular momentum is transferred to the atoms by a stimulated Raman transition, this angular momentum being able to reach about ten $\hbar$ per atom. Once created, this permanent current has a long life-time. The graph shows that an initial angular momentum of $3\hbar$ /atom can persist for about a minute. This current is measured from the size of the central hole in the cloud in a time-of-flight experiment. These figures are taken from Moulder, Beattie, et al. (2012) (group of Z. Hadzibabic at Cambridge). Similar experiments are being carried out in the group of G. Campbell at NIST.

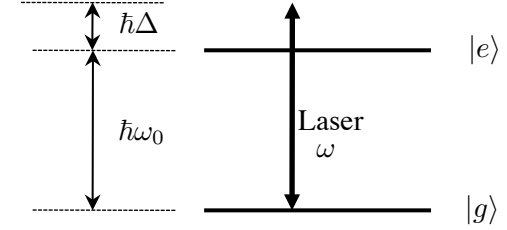


Figure IV.7. Modeling an atom as a two-level system.

The Hamiltonian describing the internal dynamics of the atom is the sum of $\hat{H}_0$ and the atom-laser coupling $\hat{V}_{\text{AL}}(t)$:

$$\hat{H}_{\text{interne}}(t) = \hat{H}_0 + \hat{V}_{\text{AL}}(t). \quad (\text{IV.45})$$

The form of this coupling can be simplified using the rotating-wave approximation², to be written (Cohen-Tannoudji, Dupont-Roc, et al. 1989):

$$\hat{V}_{\text{AL}}(t) = \frac{\hbar\kappa}{2} e^{-i\omega t} |e\rangle\langle g| + \frac{\hbar\kappa^*}{2} e^{i\omega t} |g\rangle\langle e|. \quad (\text{IV.46})$$

where the complex *Rabi frequency* κ characterizes the strength of the atom–light coupling, which we will here assume to be of electric-dipole nature. In particular, the term $(\hbar\kappa/2) e^{-i\omega t} |e\rangle\langle g|$ describes the transition from $|g\rangle$ to $|e\rangle$ by *absorption* of a photon from the incident light wave, and the term $(\hbar\kappa^*/2) e^{i\omega t} |g\rangle\langle e|$ describes the transition from $|e\rangle$ to $|g\rangle$ by *stimulated emission* of a photon into this wave.

Going into the rotating frame with respect to the internal variables makes

²This approximation consists in neglecting the terms that would contribute as $e^{\pm i(\omega+\omega_0)t}$, keeping only the terms in $e^{\pm i(\omega-\omega_0)t}$ (cf. 2012-13 course).

it possible to reduce the problem to a time-independent Hamiltonian³:

$$\hat{H}_{\text{interne}} = \hat{H}_0 + \hat{V}_{\text{AL}} \quad (\text{IV.49})$$

with

$$\hat{H}_0 = \frac{\hbar\Delta}{2} (|g\rangle\langle g| - |e\rangle\langle e|), \quad \hat{V}_{\text{AL}} = \frac{\hbar\kappa}{2} |e\rangle\langle g| + \frac{\hbar\kappa^*}{2} |g\rangle\langle e|. \quad (\text{IV.50})$$

i.e., in matrix form in the basis $\{|g\rangle, |e\rangle\}$:

$$\hat{H}_{\text{interne}} = \frac{\hbar}{2} \begin{pmatrix} \Delta & \kappa^* \\ \kappa & -\Delta \end{pmatrix}. \quad (\text{IV.51})$$

It is this Hamiltonian $\hat{H}_{\text{interne}}$ that we will use in what follows, omitting the sign $\sim$ to simplify the notation when there is no ambiguity. The intensity and phase characteristics of the light beam are contained in the expression for the complex Rabi frequency κ . In particular, when we study the motion of the center of mass of the atom, κ will be a function of the position of this center of mass, denoted $\kappa(\mathbf{r})$. For example, for an atom illuminated by a traveling plane wave, we will have:

$$\text{Plane laser wave with wave vector } \mathbf{k}: \quad \kappa(\mathbf{r}) = \kappa_0 e^{i\mathbf{k}\cdot\mathbf{r}}, \quad (\text{IV.52})$$

where the amplitude κ_0 will be proportional to the electric field of the laser wave, and hence to the square root of its intensity.

To be useful for the problem of interest to us here, namely the generation of artificial gauge fields, this coupling between the ground electronic state and the excited electronic state must be achievable without spontaneous emission of photons. Such spontaneous-emission processes are indeed incoherent and constitute sources of heating for the atomic gas one

³This transition is carried out via the unitary transformation associated with the operator

$$\hat{U}(t) = \exp[i(|e\rangle\langle e| - |g\rangle\langle g|)\omega t/2] \quad (\text{IV.47})$$

and the transformed Hamiltonian (here time-independent) is obtained via:

$$\hat{H} = \hat{U}(t)\hat{H}(t)\hat{U}^\dagger(t) + i\hbar\frac{d\hat{U}(t)}{dt}\hat{U}^\dagger(t). \quad (\text{IV.48})$$

seeks to manipulate. The use of an excited electronic state is thus only possible for classes of atoms for which there exist excited states with very long lifetimes, much longer than the typical duration of an experiment.

In practice, suitable atoms are atoms with two external electrons, such as Mg, Ca, Sr, Hg, Yb: their ground state is a spin-singlet state, and certain excited states can then be metastable, if they correspond to a spin triplet. The Ytterbium atom, for example, has a ground state $6s6s\ ^1S_0$ and possesses an excited state $6s6p\ ^3P_0$ with a lifetime estimated at more than 10 seconds, which is perfectly suited to this type of experiment, at least at densities low enough for inelastic collisions involving atoms in the excited state to be negligible [for a study of these collisions, see for example Scazza, Hofrichter, et al. (2014)]. The use of optical lattices, in which the probability of having more than one atom per site can be made very small, also makes it possible to circumvent this difficulty related to inelastic collisions.

3-2 The model of a Raman transition

We considered in the paragraph above a transition between the ground state $|g\rangle$ and an excited state $|e\rangle$, coupled to $|g\rangle$ by an electric-dipole transition. For most atomic species, in particular alkali atoms, states $|e\rangle$ of this type have a spontaneous-emission lifetime of the order of a few tens or a few hundreds of nanoseconds. This lifetime is far too short for the transition $|g\rangle \leftrightarrow |e\rangle$ to be usable as is to generate a usable artificial magnetic field.

This difficulty can be circumvented by taking advantage of the fact that the ground level is itself made up of several sublevels, and by using Raman transitions between these sublevels (figure V.1). Consider two sublevels $|g_1\rangle$ and $|g_2\rangle$, separated by the energy $E_2 - E_1$, and suppose that the atom is illuminated by two light waves with respective angular frequencies ω_a and ω_b , with:

$$\hbar(\omega_a - \omega_b) \approx E_2 - E_1. \quad (\text{IV.53})$$

The two levels $|g_1\rangle$ and $|g_2\rangle$ can be coherently coupled by means of a transition involving one (or more) excited electronic states. In the case where only a single state $|e\rangle$ is involved, the coupling can be written (after going

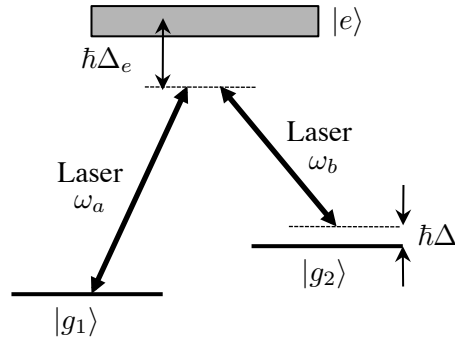


Figure IV.8. Excitation of a Raman transition between two ground sublevels $|g_1\rangle$ and $|g_2\rangle$.

into the rotating frame):

$$\hat{V}_{AL} = \frac{\hbar\kappa_a}{2}|e\rangle\langle g_1| + \frac{\hbar\kappa_b}{2}|e\rangle\langle g_2| + \text{h.c.} \quad (\text{IV.54})$$

The excited state can then be eliminated to reduce the problem to an effective Hamiltonian acting in the subspace spanned by the two states $|g_1\rangle$ and $|g_2\rangle$. For example, the coupling allowing a transition from $|g_1\rangle$ to $|g_2\rangle$, by absorption of a photon from wave a , followed by stimulated emission of a photon into wave b , is given to lowest order by the effective Rabi frequency:

$$\frac{\hbar\kappa_{\text{eff}}}{2} = \left(\frac{\hbar\kappa_a}{2}\right)_{g_1 \leftrightarrow e} \times \frac{1}{\hbar\Delta_e} \times \left(\frac{\hbar\kappa_b^*}{2}\right)_{e \leftrightarrow g_2} \quad (\text{IV.55})$$

The Hamiltonian describing the internal dynamics of the atom in the subspace $\{|g_1\rangle, |g_2\rangle\}$ is a 2×2 matrix with a structure similar to (IV.51), where the detuning $\Delta = \omega - \omega_0$ is replaced by the Raman detuning:

$$\Delta = \hbar(\omega_a - \omega_b) - (E_2 - E_1) \quad (\text{IV.56})$$

and κ by κ_{eff} . In particular, if waves a and b are plane waves with wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$, this effective Rabi frequency is written

$$\text{Plane waves } \mathbf{k}_a, \mathbf{k}_b : \quad \kappa_{\text{eff}}(\mathbf{r}) = \kappa_0 e^{i(\mathbf{k}_a - \mathbf{k}_b) \cdot \mathbf{r}}. \quad (\text{IV.57})$$

By changing the angle between the wave vectors $\mathbf{k}_a$ and $\mathbf{k}_b$, one can adjust the momentum transfer in this Raman transition from a nearly zero value (when $\mathbf{k}_a$ and $\mathbf{k}_b$ are collinear⁴) up to $2k_0 = 2\omega_0/c$, when $\mathbf{k}_a$ and $\mathbf{k}_b$ are opposite. This flexibility in the choice of momentum transfer, at a given detuning Δ , is a very valuable feature of these Raman transitions.

Light shifts. When using a Raman transition, the previous Hamiltonian (IV.51) must be supplemented by adding to the diagonal coefficients the light shifts of the states $|g_1\rangle$ and $|g_2\rangle$. These shifts correspond to *absorption-stimulated-emission* processes starting from and returning to $|g_1\rangle$ (and likewise for $|g_2\rangle$). In the three-level atom model adopted here, they equal $|\kappa_j|^2/4\Delta_e$ ($j = 1, 2$), so that the effective Hamiltonian in the subspace $\{|g_1\rangle, |g_2\rangle\}$ is written:

$$\hat{H}_{\text{interne}} = \frac{\hbar}{2} \begin{pmatrix} \Delta + |\kappa_a|^2/2\Delta_e & \kappa_a^* \kappa_b/2\Delta_e \\ \kappa_a \kappa_b^*/2\Delta_e & -\Delta + |\kappa_b|^2/2\Delta_e \end{pmatrix}. \quad (\text{IV.58})$$

When waves a and b are plane waves, these light shifts are uniform in space and can therefore be reabsorbed into the definition of the energies E_1 and E_2 of the sublevels $|g_1\rangle$ and $|g_2\rangle$.

Let us mention a remarkable special case for a zero Raman detuning Δ . In this case, the effective Hamiltonian

$$\hat{H}_{\text{interne}} = \frac{\hbar}{4\Delta_e} \begin{pmatrix} |\kappa_a|^2 & \kappa_a^* \kappa_b \\ \kappa_a \kappa_b^* & |\kappa_b|^2 \end{pmatrix}. \quad (\text{IV.59})$$

is easy to diagonalize. In particular, one of its two eigenvalues is 0, and the associated eigenvector is the *dark state* (Arimondo 1996):

$$|\psi_{\text{noir}}\rangle \propto \kappa_b |g_1\rangle - \kappa_a |g_2\rangle, \quad (\text{IV.60})$$

which one can immediately check is not coupled to the excited state $|e\rangle$ by the coupling (IV.54).

⁴One then recovers the result that would be obtained by inducing the transition $g_1 \leftrightarrow g_2$ with a microwave.

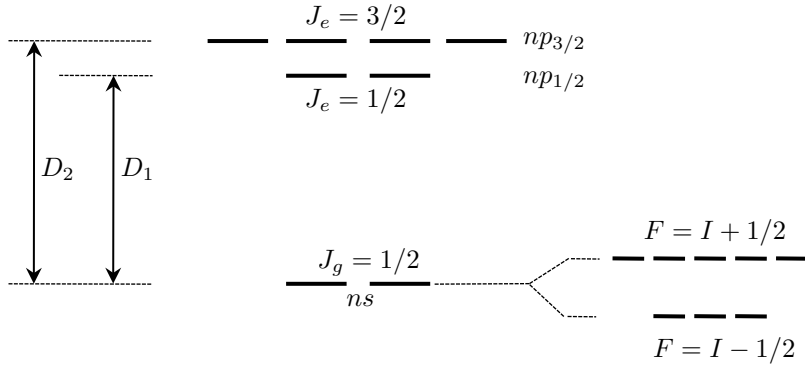


Figure IV.9. Resonance transition of an alkali atom, with the fine structure of the excited state and the hyperfine structure of the ground level, shown here for a nuclear spin $I = 3/2$ (the case of ${}^7\text{Li}$, ${}^{23}\text{Na}$, ${}^{39}\text{K}$, ${}^{41}\text{K}$, ${}^{87}\text{Rb}$). The splitting of the Zeeman sublevels shown in the figure on the right is induced by an external magnetic field.

3-3 Raman transitions: alkalis vs. lanthanides

As we have indicated, the use of a Raman transition between states derived from the ground level is a very effective way of reducing the heating and decoherence due to spontaneous-emission processes. However, this reduction is not necessarily sufficient for the “most popular” atoms in cold-atom experiments, namely alkali atoms. A much more favorable class of atoms is drawn from the lanthanide category, erbium or dysprosium for example. The results of several experiments manipulating these atoms have recently been published (Aikawa, Frisch, et al. 2012; Lu, Burdick, et al. 2012), and we will briefly explain why these atomic species are very promising compared to alkalis. To do this, we will focus on the figure of merit

$$\mathcal{M} = \frac{\kappa_{\text{eff}}}{\gamma} \quad (\text{IV.61})$$

where γ is the spontaneous-emission rate associated with the beams creating the Raman coupling. Our discussion will follow fairly closely those of Nascimbene (2013) and Cui, Lian, et al. (2013).

Alkali atoms. Consider an alkali atom, with electronic ground level $|g\rangle \equiv |ns\rangle$, illuminated by a light beam with a frequency relatively close to the atom’s resonance line $ns \leftrightarrow np$. Recall that the first excited level is actually split into two sublevels $np_{1/2}$ and $np_{3/2}$ due to the fine-structure interaction, giving rise to two resonance lines D_1 and D_2 (the famous “yellow doublet” of sodium, for example). Moreover, since the nuclear spin I is nonzero, the ground level ns and the excited levels $np_{1/2}$ and $np_{3/2}$ are also split by the hyperfine-structure interaction. In particular, the ground level splits into two sublevels of angular momentum $F = I \pm 1/2$ (figure IV.9).

Suppose that an external magnetic field is applied to lift the degeneracy between the various Zeeman states. When the atom is illuminated by two light beams with suitably chosen frequency and polarization, resonant Raman transitions can occur between adjacent Zeeman states (figure IV.9, right):

$$|g, F, m_F\rangle \leftrightarrow |g, F, m_F \pm 1\rangle. \quad (\text{IV.62})$$

The order of magnitude of the corresponding matrix element is

$$\kappa_{\text{eff}} \sim \kappa^2 \left(\frac{1}{\Delta_2} - \frac{1}{\Delta_1} \right) \quad (\text{IV.63})$$

where $\Delta_1 = \omega - \omega_1$ and $\Delta_2 = \omega - \omega_2$ are the laser detunings from the D_1 and D_2 resonance transitions. The contributions of these two transitions appear with opposite signs in (IV.63), which corresponds to destructive interference between the two paths going from $|g, F, m_F\rangle$ to $|g, F, m_F \pm 1\rangle$, via $np_{1/2}$ or $np_{3/2}$.

Let us recall the origin of this phenomenon, discussed in more detail in last year’s course. The atom-light coupling is of electric-dipole type, $q\mathbf{r} \cdot \mathcal{E}$, where $\mathbf{r}$ is the position of the outer electron of the atom and $\mathcal{E}$ is the light electric field. This coupling thus acts only on the orbital variables of the electron and does not involve its spin. The fact that the electron spin can nevertheless be flipped in a Raman transition comes from the spin-orbit coupling responsible for lifting the degeneracy between $np_{1/2}$ and $np_{3/2}$. Now, when one takes a light detuning large compared to the fine structure ($\Delta_1 \approx \Delta_2$), the effect of this spin-orbit coupling becomes negligible, and it is natural that the Raman coupling should quickly tend to 0 (as $1/\Delta^2$), faster than one might naively have assumed from (IV.55).

The spontaneous-emission rate for an alkali atom, on the other hand, is (up to numerical factors)

$$\gamma \sim \Gamma \kappa^2 \left(\frac{1}{\Delta_1^2} + \frac{1}{\Delta_2^2} \right), \quad (\text{IV.64})$$

where Γ^{-1} denotes the radiative lifetime of the excited levels $np_{1/2}$ and $np_{3/2}$. The Raman matrix element and the spontaneous-emission rate are thus both proportional to the light intensity (κ^2), so that the figure of merit $\mathcal{M}$ does not depend on this intensity, but simply on the detuning. The optimal figure of merit is obtained by taking a laser beam such that $\Delta_1 \approx -\Delta_2$, which leads to

$$\mathcal{M} \sim \frac{\Delta_{\text{s.f.}}}{\Gamma}, \quad (\text{IV.65})$$

where $\Delta_{\text{s.f.}} = \Delta_1 - \Delta_2$ represents the fine-structure splitting between the levels $np_{1/2}$ and $np_{3/2}$.

For the rubidium atom, once the numerical factors are correctly taken into account, this figure of merit is about $1.5 \cdot 10^5$, which corresponds to heating of about fifteen nanokelvins per second if the Raman matrix element is taken equal to the recoil energy $\hbar^2 k^2 / 2M$ (the minimal value for generating gauge fields within the adiabatic approximation). If one wishes to work with fermionic atoms, one must turn to potassium 40; the figure of merit is then much smaller, and the heating rate for $\hbar \kappa_{\text{eff}} = E_r$ reaches 700 nK/s (Nascimbene 2013). These heating rates are notably too large for research into strongly correlated states; indeed, we will see that the energies characterizing these states are themselves of order ten nanokelvins, and the times needed to reach thermodynamic equilibrium can exceed one second.

Lanthanides: erbium, dysprosium. Let us now consider atomic species such as erbium and dysprosium. The electronic properties of these atoms are very different from those of alkalis. To some extent, these atoms can be regarded as having two external electrons (level $6s^2$ for Dy), but their ground state does not have zero orbital angular momentum, owing to the electrons arranged in the last incomplete inner shell ($4f^{10}$ for Dy, leading to an orbital angular momentum $L = 6$). The low-lying part of the energy spectrum of these atoms thus contains lines corresponding to the excitation of one of the two external electrons or of an electron from this inner shell.

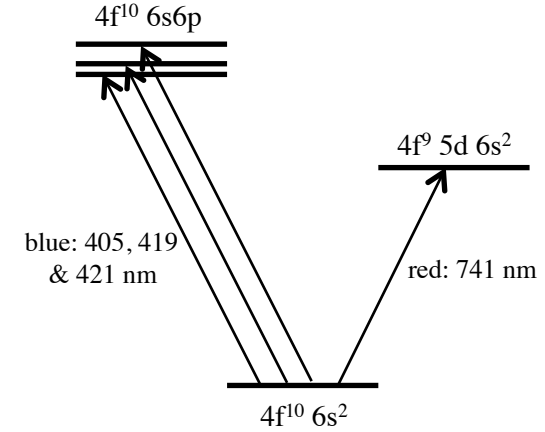


Figure IV.10. A few relevant levels of Dysprosium: the main resonance line, split by the fine structure, is in the blue. An efficient Raman coupling between Zeeman states of the ground state is obtained by taking a laser close to resonance with a narrow line located in the red, for example the line at 741 nm.

We have reported in figure IV.10 a few relevant levels of dysprosium [for erbium, a detailed study is presented by Lepers, Wyart, et al. (2014)]. One first finds an intense blue line, corresponding to the resonance transition $6s^2 \rightarrow 6s6p$ with natural width $\Gamma_b/2\pi = 30$ MHz. This line is split by the fine-structure interaction into three components with wavelengths 405, 419, and 421 nm. If this line were the only one present, one would encounter the same problem as for alkali atoms in generating a large Raman matrix element: when the laser detuning is large compared to the spacing between these lines, the matrix element decreases as the square of the detuning, and the figure of merit $\mathcal{M}$ would not be very good. But there are also many narrow lines in the atomic spectrum of dysprosium, such as the red line corresponding to the excitation of an electron from the inner shell $4f^{10} \rightarrow 4f^9 5d$ at 741 nm (term $^5K^\circ$), with a width $\Gamma_r/2\pi \sim 2$ kHz (Cui, Lian, et al. 2013).

The idea is then to use a Raman transition close to resonance with this narrow level, with a detuning that optimizes the figure of merit. More

precisely, the Raman coupling is of order

$$\kappa_{\text{eff}} \sim \frac{(d_r \mathcal{E})^2}{\Delta_r} \quad (\text{IV.66})$$

(where the subscript r refers to the *red* transition), while the spontaneous-emission rate is obtained by summing the contributions of the red and blue lines⁵:

$$\gamma \sim \Gamma_r \frac{(d_r \mathcal{E})^2}{\Delta_r^2} + \Gamma_b \frac{(d_b \mathcal{E})^2}{\Delta_b^2}. \quad (\text{IV.67})$$

In these expressions, we have introduced the atomic dipoles d_b and d_r associated with the blue and red transitions, with

$$\frac{\Gamma_r}{\Gamma_b} \sim \frac{d_r^2}{d_b^2}. \quad (\text{IV.68})$$

The optimization of the figure of merit is then simple and leads to a detuning such that

$$|\Delta_r| \sim |\Delta_b| \frac{\Gamma_r}{\Gamma_b}. \quad (\text{IV.69})$$

Since the ratio Γ_r/Γ_b is very small compared to 1 ($\sim 10^{-4}$), this amounts to choosing a laser for the Raman coupling detuned by about a tenth of a nanometer from the red line. The figure of merit is then simply

$$\mathcal{M} \sim \frac{\Delta_b}{\Gamma_b}, \quad (\text{IV.70})$$

which is much larger than the result for alkalis, since Δ_b is of the order of an optical frequency. Once all the numerical factors are taken into account, one finds a heating rate below one nanokelvin per second for a Raman coupling equal to the recoil energy, for both bosonic and fermionic isotopes; this appears markedly more promising than the case of alkali atoms (Nascimbene 2013).

⁵Here we give expressions of the “rotating-wave approximation” type, whereas for the large detunings involved for the blue transition, the anti-resonant terms can make a significant contribution. We can therefore only hope to obtain here the order of magnitude of the figure of merit. A more precise calculation is of course possible, at the cost of less transparent mathematical expressions.

4 Vector and scalar potentials for an atom

The first transposition of the concepts of geometric gauge potentials to the case of an atom moving in a laser wave is due to Dum & Olshanii (1996), who were interested in the motion of an atom in an optical lattice. They established the general expressions for the gauge potentials, and then focused on the case of the scalar potential: a key point of their approach is that a “dark” state, a linear combination of internal states in principle uncoupled from the light, nevertheless experiences a nonzero force due to the gradient of the geometric scalar potential.

In this paragraph, we will begin by establishing a general expression for these potentials, and then specialize our discussion to the simplest situation, the case of a two-level atom. We will conclude with a description of the first experiment to demonstrate the geometric scalar potential on an atom, carried out by Dutta, Teo, et al. (1999) as a follow-up to the proposal of Dum & Olshanii (1996).

4-1 General expressions

In the previous paragraph, we cast the expression for the internal Hamiltonian of the atom coupled to the radiation in the form of a $(2F+1) \times (2F+1)$ matrix. For example, for a two-level atom, we wrote this Hamiltonian in the basis $\{|g\rangle, |e\rangle\}$ in the form

$$\hat{H}_{\text{interne}} = \frac{\hbar}{2} \begin{pmatrix} \Delta & \kappa^* \\ \kappa & -\Delta \end{pmatrix}. \quad (\text{IV.71})$$

This matrix can *a priori* depend on the position of the center of mass of the atom through several parameters:

- the phase of the laser beam(s) incident on the atom, which appears in the phase of the coefficient κ ,
- the intensity of the laser beam, which is proportional to $|\kappa|^2$,
- a polarization gradient of the light beam,
- a possible spatial variation of the detuning Δ , obtained for example by placing the atom in an inhomogeneous magnetic field.

The general procedure we followed in the previous chapter to bring out the notion of Berry connection applies as it stands here. For any position $\mathbf{r}$ of the center of mass, one can diagonalize the matrix $\hat{H}_{\text{interne}}$ and obtain its $2F + 1$ eigenstates $|\psi_n(\mathbf{r})\rangle$, with energies denoted $E_n(\mathbf{r})$, which we will call *dressed states*. Suppose the atom is prepared at the initial time $t = 0$ in one of these dressed states:

$$\Psi(\mathbf{r}, 0) = \phi_\ell(\mathbf{r}, 0) |\psi_\ell(\mathbf{r})\rangle. \quad (\text{IV.72})$$

If the conditions necessary for the adiabatic approximation are satisfied, and if the spontaneous-emission processes can be neglected, the atom will adiabatically follow this dressed state, and we will have at every instant:

$$\Psi(\mathbf{r}, t) = \phi_\ell(\mathbf{r}, t) |\psi_\ell(\mathbf{r})\rangle. \quad (\text{IV.73})$$

We saw in the previous chapter that the external motion of the atom, described by the evolution of the wave function $\phi_\ell(\mathbf{r}, t)$, is given by a Schrödinger equation for a scalar particle (with no internal structure) subject to the vector potential (Berry connection):

$$\mathcal{A}_\ell(\mathbf{r}) = i\hbar \langle \psi_\ell | \nabla \psi_\ell \rangle, \quad (\text{IV.74})$$

this effective Schrödinger equation being written here

$$i\hbar \frac{\partial \phi_\ell}{\partial t} = \left[\frac{(\hat{\mathbf{p}} - \mathcal{A}_\ell(\mathbf{r}))^2}{2M} + E_\ell(\mathbf{r}) + \mathcal{V}_\ell(\mathbf{r}) \right] \phi_\ell(\mathbf{r}, t), \quad (\text{IV.75})$$

where $\mathcal{V}_\ell(\mathbf{r})$ is the additional scalar potential, which also appears due to the adiabatic approximation:

$$\mathcal{V}_\ell(\mathbf{r}) = \frac{\hbar^2}{2M} \sum_{n \neq \ell} \langle \psi_n | \nabla \psi_\ell \rangle|^2. \quad (\text{IV.76})$$

4-2 The result for a two-level atom

We can make the results (IV.74) and (IV.76) explicit for the case of the two-level atom by casting the 2×2 matrix given in (IV.71) in the form

$$\hat{H}_{\text{interne}} = \frac{\hbar\Omega}{2} \begin{pmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -\cos \theta \end{pmatrix}, \quad (\text{IV.77})$$

with, for the generalized Rabi frequency Ω , the mixing angle θ , and the phase angle ϕ :

$$\Omega = \sqrt{\Delta^2 + |\kappa|^2}, \quad \cos \theta = \frac{\Delta}{\Omega}, \quad \sin \theta = \frac{|\kappa|}{\Omega}, \quad \kappa = |\kappa| e^{i\phi}, \quad (\text{IV.78})$$

where Ω , θ and ϕ are functions of $\mathbf{r}$, with $\theta \in [0, \pi]$. The two dressed states are then

$$|\psi_+\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \end{pmatrix}, \quad |\psi_-\rangle = \begin{pmatrix} \sin(\theta/2) \\ -e^{i\phi} \cos(\theta/2) \end{pmatrix}, \quad (\text{IV.79})$$

with the associated energies:

$$E_\pm = \pm \frac{\hbar\Omega}{2}. \quad (\text{IV.80})$$

The Berry connection for each of these states is easily computed and equals

$$\mathcal{A}_\pm(\mathbf{r}) = \frac{\hbar}{2} (-1 \pm \cos \theta) \nabla \phi. \quad (\text{IV.81})$$

The light wave must therefore have a phase gradient in order to give rise to a nonzero geometric vector potential. In particular, a standing light wave always leads to $\mathcal{A} = 0$.

The scalar potential, in turn, is given by

$$\mathcal{V}_\pm(\mathbf{r}) = \frac{\hbar^2}{8M} [(\nabla \theta)^2 + \sin^2 \theta (\nabla \phi)^2], \quad (\text{IV.82})$$

and it creates a potential bump at the points where the dressed states vary rapidly with the position $\mathbf{r}$.

Remark: gauge transformation. The value of $\mathcal{A}_\pm$ depends on the choice of gauge, i.e., on the local phase chosen to write the states $|\psi_\pm\rangle$. In (V.8), we adopted a convention that allows an easy match (with no additional phase) to the state $|g\rangle$. Another “natural” choice consists instead in favoring the match to $|e\rangle$, and thus taking

$$|\psi'_+\rangle = \begin{pmatrix} e^{-i\phi} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}, \quad |\psi'_-\rangle = \begin{pmatrix} e^{i\phi} \sin(\theta/2) \\ -\cos(\theta/2) \end{pmatrix} \quad (\text{IV.83})$$

which leads to

$$\mathcal{A}'_{\pm}(\mathbf{r}) = \frac{\hbar}{2} (1 \pm \cos \theta) \nabla \phi. \quad (\text{IV.84})$$

One can also mix and match, taking (for example) the value (V.8) for $|\psi_+\rangle$ and the value (IV.83) for $|\psi'_-\rangle$... In other words, gauge invariance manifests itself here through the possibility of adding to, or subtracting from, $\mathcal{A}_{\pm}$ a term proportional only to $\nabla \phi$. The scalar potential, for its part, is gauge invariant.

4-3 The scalar potential in a “gray” lattice

To demonstrate the scalar potential $\mathcal{V}$, Dutta, Teo, et al. (1999) used a one-dimensional optical lattice. Following the initial idea of Dum & Olshanii (1996), they chose a configuration in which the dressed level $E_1(\mathbf{r})$ was nearly flat, so that the force due to the geometric scalar potential $\mathcal{V}_1(\mathbf{r})$ would constitute a significant modification of the atoms’ dynamics.

The 1D lattice consists of two counter-propagating light waves along the z axis. These two waves have crossed linear polarizations ϵ_x and ϵ_y (figure IV.11a). When the electric fields of these two waves are coherently added, one obtains a constant intensity, but a resultant polarization that varies along the z axis. At certain points, this polarization is linear and oriented at 45° relative to the x, y axes. At a distance $\lambda/8$ from these points, the polarization is right or left circular. Between these particular points, the polarization is generally elliptical.

The laser is chosen nearly resonant with the D_1 line transition of the rubidium atom (isotope 87), more precisely with the transition connecting $|g, F = 2\rangle$ and $|e, F = 2\rangle$ (figure IV.11b). If this were the only transition involved, one can show that one of the dressed states would be “dark” at every point in space, i.e., that the energy $E_1(z)$ of the dressed level $|\psi_1(z)\rangle$ would be zero. This can be simply understood at a few particular points on the z axis:

- Where the polarization is circular $\sigma_{\pm}$, the state $|\psi_1(z)\rangle$ corresponding to $E_1 = 0$ is the polarized state $|g, F = 2, m_z = \pm 2\rangle$.
- Where the polarization is linear, one can take the axis u of this polarization as the quantization axis, and the state $|\psi_1(z)\rangle$ then corresponds

to $|g, F = 2, m_u = 0\rangle$. Indeed, for integer F , the Clebsch-Gordan coefficient connecting the two states $|g, F, m = 0\rangle$ and $|e, F, m = 0\rangle$ is always zero.

Consequently, if this transition $|g, F = 2\rangle$ and $|e, F = 2\rangle$ were the only one involved, one would deduce that $E_1(z) = 0$ for all z , which means that the dynamics would be due solely to the scalar potential $\mathcal{V}(z)$: its identification would be easy.

The proximity of the excited level $|e, F = 1\rangle$ slightly perturbs this reasoning. While the states $|g, F = 2, m_z = \pm 2\rangle$ remain dark where the light is $\sigma_{\pm}$, this is not the case at the points where the light is linearly polarized: the state $|g, F = 2, m_u = 0\rangle$ becomes slightly coupled to the light via the level $|e, F = 1, m_u = 0\rangle$. As a result, the energy $E_1(z)$ is slightly modulated along the z axis, as shown in figure IV.11c. Nevertheless, this residual modulation of the energy $E_1(z)$ is small enough that the scalar potential $\mathcal{V}(z)$ plays a very significant role for the lowest dressed level. These two functions are plotted on the same scale in figure IV.11, and they indeed have comparable amplitudes.

To study the dynamics of the atoms in this lattice, Dutta, Teo, et al. (1999) prepared the atoms by optical pumping into the wells corresponding to σ_+ polarization, these atoms being predominantly in the lowest-energy dressed level⁶. They then studied the dynamics of the transfer from the σ_+ wells to the σ_- wells due to tunneling through the barrier created by the superposition $E_1(z) + \mathcal{V}(z)$. The corresponding tunneling oscillation is shown in figure IV.11d, with a numerical simulation reproducing the experimental result remarkably well.

From this oscillation one deduces a characteristic period; Dutta, Teo, et al. (1999) measured this period for various powers of the standing wave, and their result is shown in figure IV.11e. If the potential $E_1(z)$ were the only one present, one would expect the period to increase with intensity, since the barrier height would be proportional to this intensity. But the geometric scalar potential $\mathcal{V}(z)$ is practically independent of the intensity: it is therefore this potential that dominates the tunneling dynamics at low intensity, which explains why the tunneling time plotted in figure IV.11e

⁶Dutta, Teo, et al. (1999) did not use a condensate, and relied solely on laser cooling to ensure this preparation, which did not make their task any easier..

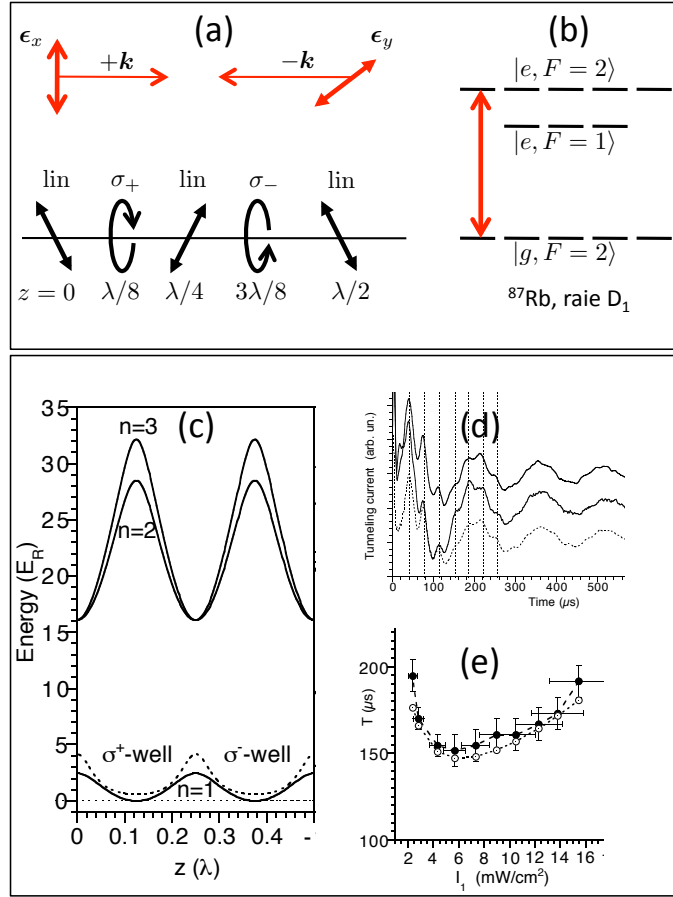


Figure IV.11. Demonstration of the scalar potential by Dutta, Teo, et al. (1999). (a) 1D laser configuration with two counter-propagating laser waves and crossed linear polarizations. The resultant polarization oscillates along the z axis, going from linear to circular with period $\lambda/2$. (b) Level scheme used: D_1 line of ^{87}Rb , with the laser quasi-resonant with the $F_g = 2 \leftrightarrow F_e = 2$ transition. (c) Variations of the position of the lowest dressed energy levels as a function of position. The scalar potential is plotted as a dotted line. (d) Tunneling transition rate from a σ_+ -polarization well to a σ_- -polarization well (theory and experiment). (e) Variation of the tunneling rate as a function of light intensity.

changes only very little when the light intensity is varied by a factor of about 5. Even though this demonstration remains qualitative and relies on fairly sophisticated comparisons with a theoretical model to be convincing, it constituted a first step in demonstrating the influence of geometric potentials on the motion of atoms.

Chapter V

Artificial magnetism and spin-orbit coupling

In this chapter we continue our exploration of the physics of geometric potentials as it can be implemented for atoms coupled to radiation. We will show here that it is indeed possible to create a nontrivial vector potential on an atomic gas, corresponding to the appearance of orbital magnetism. The proof of the existence of this orbital magnetism will be, as in the case of rotation, the observation of quantized vortices when the gas is in a superfluid regime.

We will then address another type of magnetism, corresponding to a spin-orbit coupling. This will involve replacing the magnetic coupling considered so far:

$$\hat{\mathbf{p}} \cdot \mathcal{A}(\hat{\mathbf{r}}) \quad (\text{V.1})$$

where $\mathcal{A}(\mathbf{r})$ is a static function, by a term of the type

$$\hat{\mathbf{p}} \cdot \hat{\mathbf{S}} \quad (\text{V.2})$$

where the $\hat{S}_i$, $i = x, y, z$, are the operators associated with the spin components of the atom. This spin-orbit coupling can lead to numerous original phenomena, both on the fundamental level, with the possibility of simulating new phases of matter such as topological insulators, and on the applied level, with strong analogies to spintronic devices.

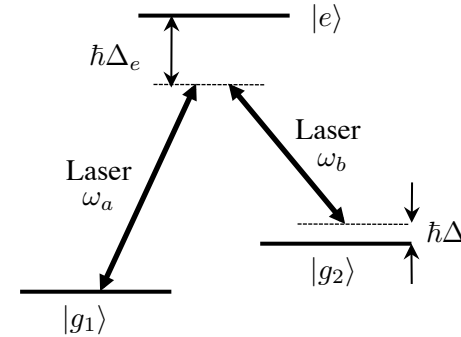


Figure V.1. Coupling of two states $|g_1\rangle$ and $|g_2\rangle$ derived from an atomic ground level, via a Raman transition. If the detuning from the excited state $|e\rangle$ is large compared to the Rabi frequencies κ_a and κ_b characterizing the coupling to the light waves, $|e\rangle$ can be perturbatively eliminated, reducing the problem to a “two-level” problem in the subspace $|g_1\rangle, |g_2\rangle$.

1 Reminder of the main concepts

1-1 Internal atomic Hamiltonian

In what follows we will consider a “two internal level” atom. This system models, for example, the configuration shown in figure V.1, in which two states $|g_1\rangle$ and $|g_2\rangle$ of the ground level have been isolated, these two states being coupled by a Raman transition involving the absorption of a photon from a laser beam starting from $|g_1\rangle$, a transition through an excited state $|e\rangle$, and stimulated emission into another beam to reach $|g_2\rangle$. As explained in the previous chapter, we assume here that the light beams are not resonant with the transitions $|g_i\rangle \leftrightarrow |e\rangle$, so that the excited state $|e\rangle$ is never significantly populated in this process. The internal dynamics of the atom can then be restricted to the two-dimensional subspace spanned by $\{|g_1\rangle, |g_2\rangle\}$.

We will denote by κ the effective Rabi frequency associated with this transition, which involves the Rabi frequencies associated with each beam as well as the detuning relative to the excited level $|e\rangle$ (cf. previous chapter):

$$\kappa = \frac{\kappa_a \kappa_b^*}{2\Delta_e}. \quad (\text{V.3})$$

The atom-laser coupling is thus written:

$$\hat{V}_{\text{AL}} = \frac{\hbar\kappa}{2}|g_2\rangle\langle g_1| + \frac{\hbar\kappa^*}{2}|g_1\rangle\langle g_2|. \quad (\text{V.4})$$

We also introduce the detuning $\hbar\Delta$ between the energy of the atomic transition $E_2 - E_1$ and the energy $\hbar(\omega_a - \omega_b)$ supplied by the light in such a transition. After eliminating the terms explicitly depending on time at the angular frequency $\omega_a - \omega_b$ (“going into the rotating frame”), the internal atomic Hamiltonian is written, in the basis $\{|g_1\rangle, |g_2\rangle\}$, in the form of a 2×2 matrix:

$$\hat{H}_{\text{interne}} = \frac{\hbar}{2} \begin{pmatrix} \Delta & \kappa^* \\ \kappa & -\Delta \end{pmatrix}. \quad (\text{V.5})$$

Recall that this matrix can *a priori* depend on the position of the center of mass of the atom through several parameters:

- the phases of the laser beams incident on the atom, which appear in the phase of the coefficient κ ,
- the light intensity, proportional to $|\kappa|^2$,
- a possible spatial variation of the detuning Δ , obtained for example by placing the atom in an inhomogeneous magnetic field.

1-2 Dressed states

We call dressed states the eigenstates of the internal Hamiltonian (V.5). To find these dressed states, it is convenient to cast the internal Hamiltonian in the form

$$\hat{H}_{\text{interne}} = \frac{\hbar\Omega}{2} \begin{pmatrix} \cos\theta & e^{-i\phi} \sin\theta \\ e^{i\phi} \sin\theta & -\cos\theta \end{pmatrix}, \quad (\text{V.6})$$

with, for the generalized Rabi frequency Ω , the mixing angle θ , and the phase angle ϕ :

$$\Omega = (\Delta^2 + |\kappa|^2)^{1/2}, \quad \cos\theta = \frac{\Delta}{\Omega}, \quad \sin\theta = \frac{|\kappa|}{\Omega}, \quad \kappa = |\kappa| e^{i\phi}, \quad (\text{V.7})$$

where Ω , θ , and ϕ are functions of $\mathbf{r}$, with $\theta \in [0, \pi]$.

The two dressed states are then (up to a phase)

$$|\psi_+\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{-i\phi} \sin(\theta/2) \end{pmatrix}, \quad |\psi_-\rangle = \begin{pmatrix} -e^{-i\phi} \sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix} \quad (\text{V.8})$$

which respectively connect to $|g_1\rangle$ and $|g_2\rangle$ as the Rabi frequency κ tends to 0, with $\Delta > 0$. The associated energies are (see figure V.2)

$$E_{\pm} = \pm \frac{\hbar\Omega}{2}. \quad (\text{V.9})$$

1-3 Geometric potentials

We now take into account the motion of the center of mass of the atom, which we describe quantum-mechanically. We will restrict our analysis

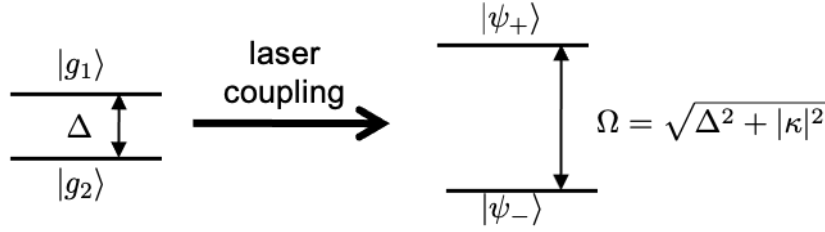


Figure V.2. Coupling between the two “bare” states (after “going into the rotating frame” at the angular frequency $\omega_a - \omega_b$), giving rise to the dressed states $|\psi_{\pm}\rangle$, linear combinations of the bare states with amplitudes depending on the point in space considered.

to the case where the atom is prepared at the initial time in one of these dressed states, for example $|\psi_-\rangle$:

$$\Psi(\mathbf{r}, 0) = \phi_-(\mathbf{r}, 0) |\psi_-(\mathbf{r})\rangle. \quad (\text{V.10})$$

We will assume that the conditions of validity of the adiabatic approximation are satisfied (we will come back to this), and the atom thus adiabatically follows this dressed state at every instant:

$$\Psi(\mathbf{r}, t) = \phi_-(\mathbf{r}, t) |\psi_-(\mathbf{r})\rangle. \quad (\text{V.11})$$

The evolution of the wave function $\phi_-(\mathbf{r}, t)$ is given by the Schrödinger equation:

$$i\hbar \frac{\partial \phi_-}{\partial t} = \left[\frac{(\hat{\mathbf{p}} - \mathcal{A}_-(\mathbf{r}))^2}{2M} + E_-(\mathbf{r}) + \mathcal{V}_-(\mathbf{r}) \right] \phi_-(\mathbf{r}, t), \quad (\text{V.12})$$

with the geometric vector and scalar potentials

$$\mathcal{A}_-(\mathbf{r}) = i\hbar \langle \psi_- | \nabla \psi_- \rangle, \quad \mathcal{V}_-(\mathbf{r}) = \frac{\hbar^2}{2M} |\langle \psi_+ | \nabla \psi_- \rangle|^2. \quad (\text{V.13})$$

Using the explicit expression for the dressed states given in (V.8), we arrive at

$$\mathcal{A}_-(\mathbf{r}) = \frac{\hbar}{2} \nabla \phi (1 - \cos \theta), \quad \mathcal{V}_-(\mathbf{r}) = \frac{\hbar^2}{8M} [(\nabla \theta)^2 + \sin^2 \theta (\nabla \phi)^2]. \quad (\text{V.14})$$

2 The case of plane waves

In this section we will consider the simplest case of two-level atoms placed in a traveling plane laser wave. To some extent, this case is trivial, since the vector potential we will induce is uniform in space. It is thus not, strictly speaking, a magnetic Hamiltonian. Nevertheless, studying this case will be instructive, since it will allow us to discuss the main momentum and energy scales of the problem, to compare our adiabatic approximation with an exact treatment, and to deduce its domain of validity. The approach we will develop will also be very useful when we turn to the spin-orbit coupling.

In the case of coupling by a plane wave, the internal Hamiltonian of each atom is the 2×2 matrix given in (V.5), with Δ independent of $\mathbf{r}$ and

$$\kappa(\mathbf{r}) = \kappa_0 e^{2i\mathbf{k} \cdot \mathbf{r}} \quad \text{with} \quad 2\mathbf{k} = \mathbf{k}_a - \mathbf{k}_b, \quad (\text{V.15})$$

so that $2\hbar\mathbf{k}$ denotes the momentum transferred to the atom by the laser beams when it undergoes a transition from $|g_1\rangle$ to $|g_2\rangle$. The Berry connection given in (V.14) is then independent of the point $\mathbf{r}$ and equals

$$\mathcal{A}_- = \hbar\mathbf{k} (1 - \cos \theta) \quad \text{with} \quad \cos \theta = \frac{\Delta}{\sqrt{\Delta^2 + \kappa_0^2}}. \quad (\text{V.16})$$

The scalar potential $\mathcal{V}_-$ given in (V.14) is also uniform in space and equal to $E_r \sin^2 \theta$, with $E_r = \hbar^2 k^2 / 2M$.

2-1 What can be measured?

Since the Berry connection (vector potential) depends on the gauge, it cannot be “measured” in any absolute sense. Nevertheless, for a given choice of gauge, it is possible¹ to obtain an experimental manifestation of its presence in a time-of-flight experiment [see also Möller & Cooper (2010)]. One way to do this is to focus on the ground state of the Hamiltonian and determine its velocity. To make this point explicit, we will compare this ground state in the absence and in the presence of laser coupling.

¹One can check that the expressions (V.21) and (V.22) are unchanged if the phase of the dressed states $|\psi_{\pm}\rangle$ is changed, which corresponds to a gauge transformation in our problem.

Let us take, for example, a detuning $\Delta > 0$ for the Raman coupling, and let us first consider a cloud of atoms prepared in the internal state $|g_2\rangle$ in the absence of the laser. The mixing angle is $\theta = 0$, and the vector potential (V.16) vanishes. The external ground state corresponds to zero momentum $\mathbf{p} = 0$, which gives for the overall (external+internal) state:

$$\Psi_0(\mathbf{r}) = |g_2\rangle. \quad (\text{V.17})$$

When the laser beams are switched on, the ground state of the internal Hamiltonian corresponds to the dressed state $|\psi_-\rangle$ given in (V.8):

$$|\psi_-(\mathbf{r})\rangle = \cos(\theta/2)|g_2\rangle - e^{-2i\mathbf{k}\cdot\mathbf{r}} \sin(\theta/2)|g_1\rangle \quad (\text{V.18})$$

and the vector potential takes the nonzero value $\mathcal{A}_- = \hbar\mathbf{k}(1 - \cos\theta)$ [cf. (V.16)]. The external Hamiltonian describing the evolution of the associated amplitude $\phi_-(\mathbf{r})$ is

$$\hat{H} = \frac{(\hat{\mathbf{p}} - \mathcal{A}_-)^2}{2m} - \frac{\hbar\Omega}{2} + \mathcal{V}_-. \quad (\text{V.19})$$

The overall (external + internal) ground state thus corresponds to the momentum $\mathbf{p} = \mathcal{A}_-$ and is written

$$\begin{aligned} \Psi(\mathbf{r}) &= e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} |\psi_-(\mathbf{r})\rangle = e^{i\mathcal{A}_-\cdot\mathbf{r}/\hbar} |\psi_-(\mathbf{r})\rangle \\ &= e^{i\mathcal{A}_-\cdot\mathbf{r}/\hbar} (\cos(\theta/2)|g_2\rangle - e^{-i2\mathbf{k}\cdot\mathbf{r}} \sin(\theta/2)|g_1\rangle). \end{aligned} \quad (\text{V.20})$$

Suppose that the laser is then abruptly switched off and that the wave packets corresponding to the two internal states $|g_1\rangle$ and $|g_2\rangle$ are allowed to propagate. Two populated velocity classes will be found, separated by the amount $2\hbar\mathbf{k}/M$:

- One velocity class has weight $\cos^2(\theta/2)$; it corresponds to the state $|g_2\rangle$ and is centered at

$$\mathbf{v} = \frac{\mathcal{A}_-}{M} = \frac{\hbar\mathbf{k}}{M}(1 - \cos\theta). \quad (\text{V.21})$$

- Another velocity class has weight $\sin^2(\theta/2)$; it corresponds to the state $|g_1\rangle$ and is centered at

$$\mathbf{v}' = \frac{\mathcal{A}_- - 2\hbar\mathbf{k}}{M} = -\frac{\hbar\mathbf{k}}{M}(1 + \cos\theta). \quad (\text{V.22})$$

Each of these velocities thus makes it possible to recover the value of $\mathcal{A}_-$ in the presence of the laser. In particular, the velocity shift of the atoms in the state $|g_2\rangle$ between the situation without laser $v = 0$ and the situation with laser $v = v_r(1 - \cos\theta)$ (cf. V.21) can reach the recoil velocity $v_r = \hbar k/M$. The maximum shift is obtained when θ approaches $\pi/2$, i.e., when the Rabi frequency κ_0 becomes large compared to the detuning Δ .

2-2 Experimental demonstration

The first experiment explicitly aimed at demonstrating a geometric vector potential for an atom in a plane light wave was carried out at NIST by Lin, Compton, et al. (2009b). Since a uniform vector potential can always be created or eliminated by a gauge transformation, one could of course debate the precedence of certain other experiments. Let us say that the experiment of Lin, Compton, et al. (2009b) was the first to frame, in terms of a gauge field, the momentum gain of an atom once it has been “dressed” by a laser. Moreover, the scheme used by Lin, Compton, et al. (2009b) lends itself well to a generalization to a nonuniform vector potential, generating a nonzero Berry curvature (magnetic field).

Lin, Compton, et al. (2009b) used ^{87}Rb rubidium atoms prepared in their hyperfine ground level $F = 1$. The atoms are placed in a static magnetic field parallel to the z axis, which we will choose as the quantization direction. The eigenstates of the atomic internal Hamiltonian in the absence of the laser are thus the three Zeeman sublevels $|m_z = -1\rangle$, $|m_z = 0\rangle$, and $|m_z = +1\rangle$, with energies $-\hbar\omega_Z$, 0 , $+\hbar\omega_Z$ (we will neglect here the second-order Zeeman effect).

A stimulated-Raman-effect coupling between these three sublevels is produced by a pair of beams (denoted a and b) counter-propagating along the x axis (figure V.3ab). Beam a , with wave vector $\mathbf{k} = k\mathbf{u}_x$, is linearly polarized along the z axis, which gives a zero angular-momentum component along this axis (π polarization). Beam b , with wave vector $-\mathbf{k}$, is linearly polarized along the y axis, which gives the photons of this beam an angular-momentum component $\pm\hbar$ along the quantization axis z (σ polarization). The two beams are chosen with frequencies ω_a and ω_b such

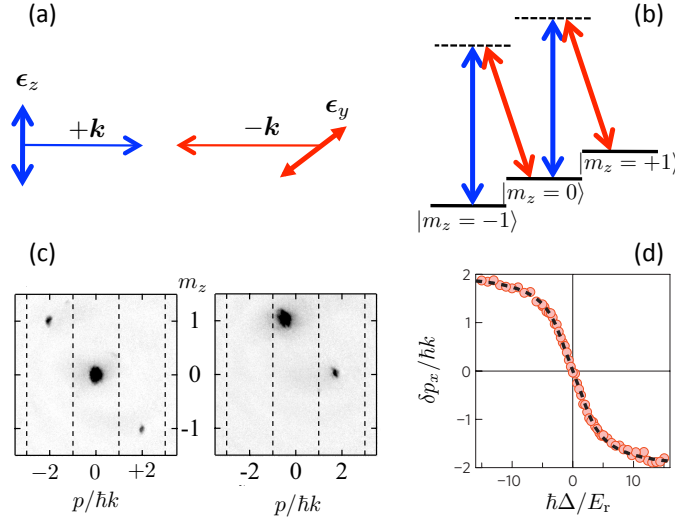


Figure V.3. (a) Laser configuration used by Lin, Compton, et al. (2009b) (b) Level scheme with the quasi-resonant laser transitions. (c) Results of a time-of-flight experiment in the presence of a magnetic-field gradient (Stern and Gerlach). (d) Momentum shift δp_x of the ground state revealing the presence of a vector potential $\mathcal{A}$ (figure taken from Lin, Compton, et al. (2011b)).

that²

$$\Delta = (\omega_a - \omega_b) - \omega_Z, \quad |\Delta| \ll \omega_Z. \quad (\text{V.25})$$

Although this is now a 3-level system instead of 2, the method followed previously still applies. Going into the rotating frame associated with this resonant approximation, the Hamiltonian of an atom interacting with the

²In practice, $\omega_Z/2\pi$ is of order 3 MHz, while $\Delta/2\pi$ is of order a few tens of kHz. In this way, only the processes

$$|m_z = -1, \mathbf{p} - 2\hbar\mathbf{k}\mathbf{u}_x\rangle \leftrightarrow |m_z = 0, \mathbf{p}\rangle \leftrightarrow |m_z = +1, \mathbf{p} + 2\hbar\mathbf{k}\mathbf{u}_x\rangle \quad (\text{V.23})$$

are quasi-resonant (up to Δ), whereas other processes such as

$$|m_z = -1, \mathbf{p} + 2\hbar\mathbf{k}\mathbf{u}_x\rangle \leftrightarrow |m_z = 0, \mathbf{p}\rangle \leftrightarrow |m_z = +1, \mathbf{p} - 2\hbar\mathbf{k}\mathbf{u}_x\rangle \quad (\text{V.24})$$

are non-resonant and can be neglected.

radiation can be written, in the basis $\{|m_z = -1\rangle, |m_z = 0\rangle, |m_z = +1\rangle\}$, as:

$$H_{\text{interne}} = \hbar \begin{pmatrix} +\Delta & \kappa_0 e^{-2ikx}/\sqrt{2} & 0 \\ \kappa_0 e^{+2ikx}/\sqrt{2} & 0 & \kappa_0 e^{-2ikx}/\sqrt{2} \\ 0 & \kappa_0 e^{+2ikx}/\sqrt{2} & -\Delta \end{pmatrix}, \quad (\text{V.26})$$

One determines the three eigenstates $|\psi_{-1}\rangle$, $|\psi_0\rangle$ and $|\psi_{+1}\rangle$ associated with the three energies $E_0 = 0$ and $E_{\pm} = \pm\hbar(\Delta^2 + \kappa_0^2)^{1/2}$. For example, one finds for the dressed state of lowest energy:

$$|\psi_{-1}\rangle = \frac{1}{2\sqrt{\Delta^2 + \kappa^2}} \begin{pmatrix} (\sqrt{\Delta^2 + \kappa^2} - \Delta) e^{-4ikx} \\ -\sqrt{2} \kappa_0 e^{-2ikx} \\ (\sqrt{\Delta^2 + \kappa^2} + \Delta) \end{pmatrix} \quad (\text{V.27})$$

from which we deduce the vector potential

$$\mathcal{A}_- = i\hbar\langle\psi_{-1}|\nabla|\psi_{-1}\rangle = 2\hbar\mathbf{k}(1 - \cos\theta) \quad \text{with} \quad \cos\theta = \frac{\Delta}{\sqrt{\Delta^2 + \kappa^2}}. \quad (\text{V.28})$$

This result is very close to the one found for a two-level atom in a plane wave, up to a factor of 2 [cf. eq.(V.16)]. This factor of 2 is due to the fact that we have gone from a situation involving an effective spin 1/2 (two-dimensional Hilbert space) to a situation with spin 1.

The experiment carried out by Lin, Compton, et al. (2009b) consisted in preparing the atoms (in practice a Bose–Einstein condensate) in the dressed state $|\psi_{-1}\rangle$ for a Raman detuning $\Delta = 0$, and then increasing the value of this detuning up to a given value. This increase is carried out slowly, so that the atoms remain in the ground state of the Hamiltonian³

$$\hat{H} = \frac{(\hat{\mathbf{p}} - \mathcal{A}_-)^2}{2M} + \dots \quad (\text{V.29})$$

A time-of-flight measurement, in which the Raman beams are abruptly switched off, then reveals the momentum components occupied by the atoms (figure V.3c). This ballistic expansion takes place in the presence of

³For this assumption to be valid, translational invariance must be broken, which is done in practice by confining the cloud in a harmonic trap. Otherwise, for an isolated atom, the momentum commutes at every instant with the Hamiltonian and is thus a constant of the motion.

a magnetic-field gradient, which makes it possible to simultaneously measure the population of the three Zeeman sublevels. One generally obtains three nonzero populations, whose weights are obtained from the expansion (V.27) in the basis of the Zeeman sublevels. Each component has a momentum shifted by the same amount δp_x relative to the value found for $\Delta = 0$. If the atoms are in the ground state of (V.29), one expects to find $\delta p_x = \mathcal{A}_-$, which is indeed the case experimentally (figure V.3d).

2-3 Exact treatment for the plane wave

In the particular case of a traveling plane laser wave, one can in fact carry out an exact quantum treatment, not requiring the use of the adiabatic approximation [see also Lin, Compton, et al. (2009b)]. If the external motion of the center of mass of the atom is taken into account quantum-mechanically, a basis of the total Hilbert space is given by $\{|g_1, \mathbf{p}\rangle, |g_2, \mathbf{p}'\rangle\}$. This infinite set of states can be grouped into families that the atom-laser interaction leaves globally invariant:

$$\mathcal{F}(\mathbf{p}) = \{|g_1, \mathbf{p} - \hbar \mathbf{k}\rangle, |g_2, \mathbf{p} + \hbar \mathbf{k}\rangle\}. \quad (\text{V.30})$$

The total Hamiltonian (internal+center of mass), restricted to the family $\mathcal{F}(\mathbf{p})$, takes the form of a 2×2 matrix:

$$\hat{H}(\mathbf{p}) = \begin{pmatrix} (\mathbf{p} - \hbar \mathbf{k})^2/2M + \hbar\Delta/2 & \hbar\kappa_0/2 \\ \hbar\kappa_0/2 & (\mathbf{p} + \hbar \mathbf{k})^2/2M - \hbar\Delta/2 \end{pmatrix} \quad (\text{V.31})$$

which can be diagonalized for any value of $\mathbf{p}$ (see figure V.4). When $\kappa_0 = 0$, the eigenenergies $E_{\pm}(\mathbf{p})$ are the two usual parabolas for the dispersion relation of a free particle. When κ_0 is nonzero, the two eigenvalues of the matrix (V.31) become:

$$E_{\pm}(\mathbf{p}) = \frac{p^2}{2M} + E_r \pm \frac{\hbar}{2} \left[\kappa_0^2 + \left(\Delta - 2 \frac{\mathbf{k} \cdot \mathbf{p}}{M} \right)^2 \right]^{1/2}, \quad (\text{V.32})$$

where we have introduced the recoil energy $E_r = \hbar^2 k^2 / 2M$.

Let us consider, for example, the lowest energy branch $E_-(\mathbf{p})$ as in paragraph § 2-1 (figure V.4). The result (V.32) indeed gives back the spectrum

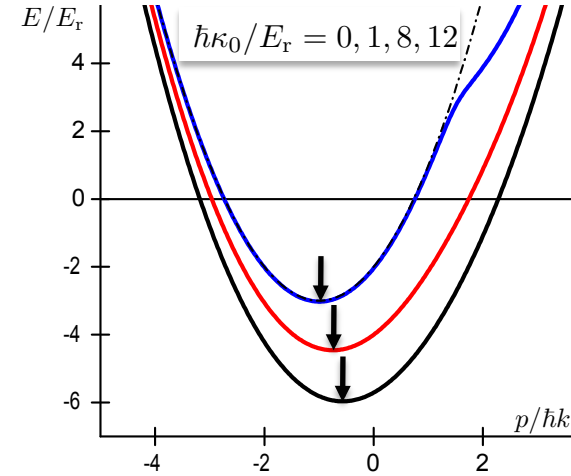


Figure V.4. Lowest energy level $E_-(\mathbf{p})$ of the Hamiltonian $\hat{H}(\mathbf{p})$ given in (V.31) for $\hbar\Delta = 6 E_r$ and for increasing values of κ_0 . The arrow marks the minimum of the dispersion curve, corresponding to the momentum $\mathbf{p} = \mathcal{A}_-$, the Berry connection $\mathcal{A}_-$ being given in (V.35).

of the Hamiltonian

$$\hat{H} = \frac{(\hat{\mathbf{p}} - \mathcal{A}_-)^2}{2M} - \frac{\hbar\Omega}{2} + \dots \quad (\text{V.33})$$

by taking the limit $kp/M \ll \Omega = (\kappa_0^2 + \Delta^2)^{1/2}$ and using $\cos \theta = \Delta/\Omega$:

$$\begin{aligned} E_-(\mathbf{p}) &\approx \frac{p^2}{2M} + E_r - \frac{\hbar\Omega}{2} \left[1 - \frac{4\Delta \mathbf{k} \cdot \mathbf{p}}{M\Omega^2} \right]^{1/2} \\ &\approx \frac{p^2}{2M} + \frac{\hbar \mathbf{k} \cdot \mathbf{p}}{2M} \cos \theta - \frac{\hbar\Omega}{2} + \dots \\ &\approx \frac{p^2}{2M} - \frac{\mathbf{p} \cdot \mathcal{A}_-}{M} - \frac{\hbar\Omega}{2} + \dots \end{aligned} \quad (\text{V.34})$$

We thus recover the result of the adiabatic approximation starting from the exact treatment. Note that we have taken here the gauge choice

$$\mathcal{A}_- = -\hbar \mathbf{k} \cos \theta, \quad (\text{V.35})$$

which differs by the constant term $\hbar \mathbf{k}$ from the result (V.16) obtained with the expression (V.18) of the dressed state $|\psi_{-}\rangle$. The gauge choice made here is obtained by writing the dressed state $|\psi_{-}\rangle$ in the form

$$|\psi_{-}\rangle = e^{i\mathbf{k}\cdot\mathbf{r}} \cos(\theta/2)|g_2\rangle - e^{-i\mathbf{k}\cdot\mathbf{r}} \sin(\theta/2)|g_1\rangle, \quad (\text{V.36})$$

a choice that corresponds to the definition of the family $\mathcal{F}(\mathbf{p})$, which gives symmetric roles to $|g_1\rangle$ and $|g_2\rangle$.

This exactly solvable case moreover allows the validity of the adiabatic approximation to be tested; recall that this validity criterion is that the angular velocity of the eigenstates must be small compared to the Bohr frequencies of the system. In the present case, this Bohr frequency is given by the generalized Rabi frequency Ω . The angular velocity of the eigenstates is obtained by studying how the motion of the atom at velocity v modifies the dressed states. Now, these dressed states vary on the scale $k^{-1} = \lambda/(2\pi)$, because of the coefficient $e^{i\mathbf{k}\cdot\mathbf{r}}$ entering their expression. The sought angular velocity, computed from a semiclassical point of view, is thus

$$\left| \langle \psi_{-} | \dot{\psi}_{-} \rangle \right| \sim kv \cos^2(\theta/2) \lesssim kv, \quad (\text{V.37})$$

which gives as a sufficient criterion for the adiabatic approximation to be satisfied

$$k\bar{v} \ll \Omega \quad \text{or equivalently} \quad \frac{kp}{M\Omega} \ll 1 \quad (\text{V.38})$$

which justifies the expansion (V.34). Since a minimal velocity scale for this problem is given by the recoil velocity $\hbar k/M$, the above criterion implies in particular

$$\frac{\hbar k^2}{M} \ll \Omega \quad \text{or equivalently} \quad E_r \ll \hbar\Omega. \quad (\text{V.39})$$

3 The Berry curvature

We now turn to the truly interesting case where not only the vector potential (Berry connection), but also the magnetic field (Berry curvature) is nonzero. The previous paragraph, in which we described how to generate a uniform vector potential, will serve as our guide. We saw that for a momentum transfer between atom and light occurring in the x direction, we

obtain a vector potential aligned with the x axis, $\mathcal{A} = \mathcal{A}u_x$ [cf. V.16], whose amplitude $\mathcal{A}$ depends on the ratio of the Rabi frequency to the detuning. The Hamiltonian is then

$$\hat{H} = \frac{(\hat{p}_x - \mathcal{A})^2}{2M} + \frac{\hat{p}_y^2}{2M} + \dots \quad (\text{V.40})$$

To obtain a genuine magnetic Hamiltonian, $\mathcal{A}$ must depend on position. For example, a vector potential $\mathcal{A} \propto y$ will allow us to realize a Hamiltonian corresponding to a magnetic field in the Landau gauge.

3-1 The general result for a two-level atom

Starting from the general expression

$$\mathcal{A} = i\hbar \langle \psi | \nabla \psi \rangle, \quad (\text{V.41})$$

where $|\psi\rangle$ is the dressed state adiabatically followed by the atom, one easily checks that the expression for the Berry curvature can be cast in the form

$$\mathcal{B} \equiv \nabla \times \mathcal{A} = i\hbar \langle \nabla \psi | \times | \nabla \psi \rangle \quad (\text{V.42})$$

For the case of the two-level atom addressed above (§1-3), this Berry curvature is written, for the two dressed levels of energy $\pm\hbar\Omega/2$,

$$\mathcal{B}_{\pm} = \nabla \times \mathcal{A}_{\pm} = \pm \frac{\hbar}{2} \nabla(\cos\theta) \times \nabla\phi. \quad (\text{V.43})$$

To have a nonzero curvature, one must have simultaneously:

- a gradient of the phase ϕ of the laser coupling (a purely standing wave will not do the job),
- a gradient of the mixing angle θ , obtained by taking
 - either a laser intensity that varies in space,
 - or a detuning depending on the point where the atom is located, which can be obtained for example by placing the atom in an external magnetic field.

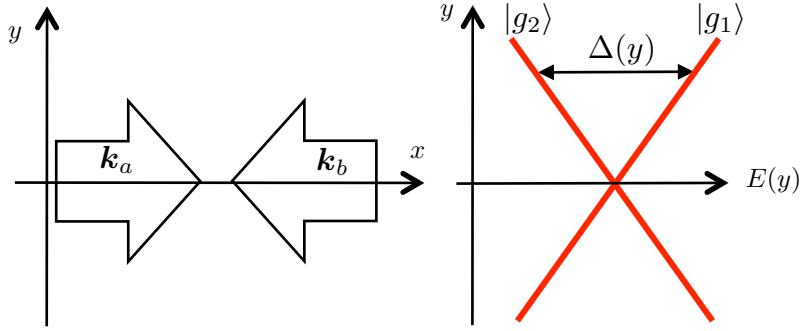


Figure V.5. Laser configuration and atomic spectrum leading to a nonzero Berry curvature.

It is interesting to note that we recover here two ingredients characteristic of the forces that a light beam can exert on a “two-level” atom. The phase gradient of the light wave is at the origin of the radiation-pressure force, while the intensity gradient of the light wave gives rise to the dipole force. To have a nonzero artificial magnetic field, these two ingredients must be present together, with the additional constraint that the two gradients must not be collinear. Note, however, that the parallel with the radiative forces is not complete. The radiation-pressure force only exists in the presence of an additional dissipative process, such as the spontaneous emission of photons. On the contrary, in the case of the artificial magnetism of interest here, no dissipative process comes into play.

3-2 Plane wave + detuning gradient

Consider a two-level atom $|g_1\rangle$ and $|g_2\rangle$, with a Raman coupling created by two plane light waves inducing a momentum transfer of $2\hbar k$ along the x axis:

$$\kappa(\mathbf{r}) = \kappa_0 e^{i2kx}, \quad (\text{V.44})$$

which gives a phase gradient $\nabla\phi = 2k\mathbf{u}_x$. Suppose that the detuning between the atom and the light wave varies linearly with position along the

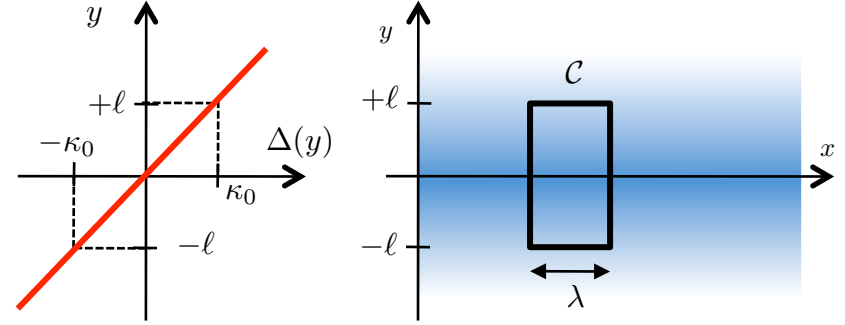


Figure V.6. False-color representation of the magnitude of the magnetic field generated by the plane wave + detuning gradient configuration. The Aharonov–Bohm–Berry phase across the contour C is of order 2π .

y axis:

$$\Delta(\mathbf{r}) = \Delta' y \quad (\text{V.45})$$

where Δ' is a constant (figure V.5). As indicated above, such a variation can be obtained using the Zeeman effect in a magnetic-field gradient (a genuinely real one, this time!) or else a light shift of the level $|g_1\rangle$ or $|g_2\rangle$ produced by an auxiliary laser beam. Let us introduce the characteristic length of the problem

$$\ell = \kappa_0 / \Delta' \quad (\text{V.46})$$

which is the distance from the y axis at which the detuning becomes (in absolute value) equal to the Rabi frequency κ_0 . A simple calculation then gives, for the Berry curvature,

$$\mathcal{B}(\mathbf{r}) = B_0 \mathcal{L}^{3/2}(y) \mathbf{u}_z \quad (\text{V.47})$$

with

$$B_0 = \frac{\hbar k}{\ell}, \quad \mathcal{L}(y) = \frac{1}{1 + y^2/\ell^2}. \quad (\text{V.48})$$

The effective magnetic field is thus invariant under translation along the x axis and maximal on this axis. It decreases on either side of this axis and tends to 0 as $|y|^{-3}$ as $y \rightarrow \infty$.

How can the amplitude of this magnetic field be measured? The amplitude B_0 gives the order of magnitude of the artificial magnetic field that we can create by this method. It is of order $\hbar$ (Planck's constant) divided by the square of a length, here $\lambda\ell$. This amplitude is not measured in Tesla; indeed, there is no electric charge to pair with B_0 . The natural way to evaluate the amplitude B_0 is to consider a closed circuit $\mathcal{C}$ and to evaluate the size this circuit must have for an Aharonov–Bohm-type phase to take a significant value ($\sim \pi$). We know that this phase $\gamma(\mathcal{C})$ is proportional to the flux of $\mathbf{B}$ through a surface S bounded by this contour:

$$\frac{\gamma(\mathcal{C})}{2\pi} = \frac{1}{\hbar} \iint_S \mathbf{B} \cdot \mathbf{u} \, d^2r, \quad (\text{V.49})$$

and it will thus be of order 2π for the circuit of area $2\lambda\ell$ shown in figure V.6. We will see in a later chapter that the flux of $\mathbf{B}$, expressed in units of $\hbar$, also gives the number of quantized vortices that are nucleated in the steady state in an interacting superfluid, localized inside the circuit $\mathcal{C}$. Large magnetic fields will thus correspond to small values of ℓ , i.e., to strong detuning gradients.

As a general rule, the maximum accessible values for these artificial fields correspond to the case where ϕ and θ both vary significantly on the scale of the wavelength, which leads to a magnetic field of order $\hbar k^2$. Another relevant criterion is to evaluate the cyclotron angular frequency. When the artificial magnetic field B_0 becomes of order $\hbar k^2$, one has

$$\hbar\omega_c = \frac{\hbar B_0}{M} \sim \frac{\hbar^2 k^2}{M} \sim E_r. \quad (\text{V.50})$$

Berry curvature and Lorentz force. It is interesting to recover, in this example, a classical concept of orbital magnetism, the Lorentz force. Let us take, for example, an atom moving along the y axis at velocity $\mathbf{v} = v_0 \mathbf{u}_y$ (i.e., with trajectory $y(t) = v_0 t$), in the absence of the artificial magnetic field (figure V.7). In the presence of this field, which is directed along z , the Lorentz force will deflect the trajectory of the particle in the xy plane. More precisely, the atom will acquire a momentum along the x axis given by

$$\Delta p_x = \int_{-\infty}^{+\infty} F_x \, dt = \int_{-\infty}^{+\infty} v_0 B_z(t) \, dt. \quad (\text{V.51})$$

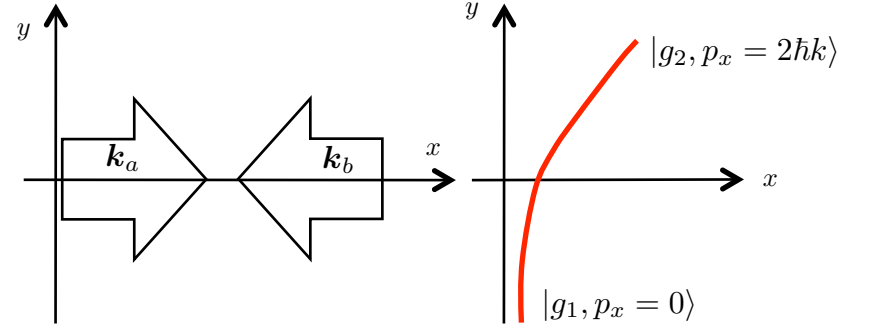


Figure V.7. Deviation of a trajectory by the Lorentz force due to the Berry curvature.

This integral can be computed analytically for the form of $\mathbf{B}$ found in (V.47), assuming that the deflection of the trajectory is small:

$$\begin{aligned} \Delta p_x &= v_0 B_0 \int_{-\infty}^{+\infty} \frac{1}{[1 + (y(t)/\ell)^2]^{3/2}} \, dt \\ &= B_0 \ell \int_{-\infty}^{+\infty} \frac{1}{[1 + u^2]^{3/2}} \, du \\ &= 2B_0 \ell \\ &= 2\hbar k. \end{aligned} \quad (\text{V.52})$$

The explanation of the deflection via the Lorentz force is thus very simple in this case: the atom enters the laser beam prepared in the dressed state that connects to $|g_1\rangle$ at $y = -\infty$; it exits after having adiabatically transitioned to the state $|g_2\rangle$, having thus absorbed the momentum $2\hbar k$ imparted by the Raman transition.

The (problematic) scalar potential. The scalar potential can also be computed simply from the general expressions given above. Using (V.14), one finds:

$$\mathcal{V}(y) = \frac{E_r}{4} \left(\mathcal{L}(y) + \frac{1}{k^2 \ell^2} \mathcal{L}^2(y) \right). \quad (\text{V.53})$$

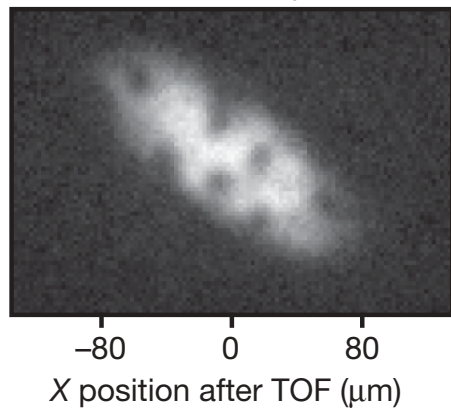


Figure V.8. Observation of vortices by Lin, Compton, et al. (2009a) in a configuration involving a pair of Raman beams that transfer momentum along one direction, and a gradient of the atomic resonance frequency along another direction.

Suppose that one seeks to achieve quantum-Hall-type effects, with the atoms accumulated in the lowest Landau level. The relevant energy scale of the problem is then $\hbar\omega_c$, which, as we saw above, is typically of order E_r . The scalar potential has an amplitude of the same order, and its influence must be taken into account precisely. An example of a detailed study is given by Julia-Diaz, Dagnino, et al. (2011); it is shown there that the strongly correlated states characteristic of the quantum Hall effect can be strongly affected, or even destroyed, by this scalar potential.

3-3 Experimental demonstration

Following on from their first experiment demonstrating the vector potential associated with a Raman transition, the NIST group made this vector potential “nontrivial,” by adding a magnetic-field gradient as indicated above. This gradient is of order a few gauss/cm, corresponding to ℓ of a few tens of the optical wavelength λ (Lin, Compton, et al. 2009a). This allowed them to observe the formation of quantized vortices, which, as we already indicated in the case of rotation, are the natural manifestation of

orbital magnetism for a superfluid. An example of a result obtained by Lin, Compton, et al. (2009a) is given in figure V.8.

The vortices observed by the NIST group, while constituting unmistakable proof of the presence of a nonzero Berry curvature, do not display the spatial order expected for an Abrikosov lattice. Lin, Compton, et al. (2009a) attribute this lack of order to heating related to spontaneous-emission processes. Recall that for alkali atoms, these processes are difficult to avoid: the Raman transitions necessary for the emergence of a gauge field only have a significant amplitude if the lasers inducing them are not too far detuned from the fine-structure splitting of the resonance line, and this small detuning results in relatively significant spontaneous emission.

4 Non-Abelian gauge fields

We now address an extension of the notion of the geometric gauge potential, and we seek to determine under what conditions one can use the notion of adiabatic following in the atom-light coupling to generate a single-particle Hamiltonian of the type

$$\hat{H} = \frac{(\hat{\mathbf{p}} - \hat{\mathcal{A}}(\hat{\mathbf{r}}))^2}{2M} + \dots \quad (\text{V.54})$$

where $\hat{\mathcal{A}}(\mathbf{r})$ is no longer simply a vector field, but a set of three matrices $\hat{\mathcal{A}} = (\hat{\mathcal{A}}_x, \hat{\mathcal{A}}_y, \hat{\mathcal{A}}_z)$ that also act in the internal Hilbert space of the atom:

$$\hat{\mathcal{A}}_x(\mathbf{r}) = [\mathcal{A}_x^{(m,n)}(\mathbf{r})], \quad \hat{\mathcal{A}}_y(\mathbf{r}) = [\mathcal{A}_y^{(m,n)}(\mathbf{r})], \quad \hat{\mathcal{A}}_z(\mathbf{r}) = [\mathcal{A}_z^{(m,n)}(\mathbf{r})], \quad (\text{V.55})$$

where the indices m, n label a basis of the internal Hilbert space.

The non-Abelian character is related to the fact that, at a given point $\mathbf{r}$, two components of this matrix generally do not commute:

$$[\hat{\mathcal{A}}_x(\mathbf{r}), \hat{\mathcal{A}}_y(\mathbf{r})] \neq 0. \quad (\text{V.56})$$

One can intuitively understand that this non-Abelian character can enrich the phenomena likely to appear in an Aharonov-Bohm-type experiment, for example. In the Abelian case, the presence of a magnetic field causes

the particle to acquire a geometric phase as it traverses a closed trajectory $\mathcal{C}$. In the non-Abelian case, this phase is replaced by a matrix acting in the internal space: the atom, even if it traverses the loop $\mathcal{C}$ at an arbitrarily slow speed, will not find itself in the same internal state at the end of this loop.

The idea of generating non-Abelian potentials geometrically dates back to the article by Wilczek & Zee (1984). In this work, the authors are interested in a generalization of the adiabatic theorem to the case where the Hamiltonian of the system possesses a group of eigenstates that remain, during the evolution, degenerate (or nearly degenerate) and well separated from the other states. This analysis was followed by applications in numerous fields ranging from molecular physics to condensed matter (Bohm, Mostafazadeh, et al. 2003; Xiao, Chang, et al. 2010) [see also Goldman, Juzeliunas, et al. (2014)].

4-1 The Lorentz force in the non-Abelian case

To properly discern the class of non-Abelian gauge fields that it is interesting to generate, we will return to the Lorentz force acting on a particle in motion when the Hamiltonian of this particle is written

$$\hat{H} = \frac{(\hat{\mathbf{p}} - \hat{\mathcal{A}}(\hat{\mathbf{r}}))^2}{2M}, \quad (\text{V.57})$$

and $\hat{\mathcal{A}}$ is an operator acting on the internal state of the particle. For this, we work in the Heisenberg picture and define the *velocity* operator as the time derivative of the position operator:

$$\hat{\mathbf{v}} = \frac{d\hat{\mathbf{r}}}{dt} = \frac{i}{\hbar} [\hat{H}, \hat{\mathbf{r}}] = \frac{\hat{\mathbf{p}} - \hat{\mathcal{A}}(\hat{\mathbf{r}})}{M}. \quad (\text{V.58})$$

The Hamiltonian (V.57) can thus be written, in the two-dimensional case of interest to us here:

$$\hat{H} = \frac{M}{2} (\hat{v}_x^2 + \hat{v}_y^2). \quad (\text{V.59})$$

Note that the velocity operator here acts both on the external degrees of freedom of the particle, via $\hat{\mathbf{p}}$, and on its internal degrees of freedom, via the matrix $\hat{\mathcal{A}}$.

We then define the *force* operator acting on the particle as the product of the mass by the derivative of the velocity operator:

$$\hat{\mathbf{F}} = M \frac{d\hat{\mathbf{v}}}{dt} = \frac{i}{\hbar} [\hat{H}, M\hat{\mathbf{v}}]. \quad (\text{V.60})$$

A simple calculation using the form (V.59) gives:

$$\hat{\mathbf{F}} = \frac{1}{2} (\hat{\mathbf{v}} \times \hat{\mathbf{B}} - \hat{\mathbf{B}} \times \hat{\mathbf{v}}) \quad (\text{V.61})$$

where the magnetic-field operator is defined by

$$\hat{\mathbf{B}} = \hat{\mathcal{B}}_z \mathbf{u}_z, \quad \hat{\mathcal{B}}_z = \frac{\partial \hat{\mathcal{A}}_x}{\partial y} - \frac{\partial \hat{\mathcal{A}}_y}{\partial x} - \frac{i}{\hbar} [\mathcal{A}_x, \mathcal{A}_y]. \quad (\text{V.62})$$

This result is a generalization of the Lorentz force; in the Abelian case, the components $\mathcal{A}_x$ and $\mathcal{A}_y$ commute and the magnetic field entering the Lorentz force (V.61) is simply the curl of the vector potential. In the non-Abelian case, an additional contribution to the magnetic field appears, proportional to the commutator $[\mathcal{A}_x, \mathcal{A}_y]$.

This commutator makes it possible to generate interesting magnetic effects, related to a nonzero Lorentz force, even when the vector potential is uniform. This is precisely what occurs in the spin-orbit coupling that we will encounter further on. We will then have:

$$\hat{\mathcal{A}} = \eta \hat{\mathbf{S}} \quad (\text{V.63})$$

where $\hat{\mathbf{S}}$ is the spin operator of the particle and η a constant. The generic commutator for an angular-momentum operator, $[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z$, will lead to a uniform magnetic-field operator

$$\hat{\mathbf{B}} = \eta^2 \hat{S}_z \mathbf{u}_z. \quad (\text{V.64})$$

The particle will experience a nonzero magnetic field aligned along the z axis, whose amplitude $\eta^2 m \hbar$ depends on the quantum number m giving the projection of its spin onto the z axis.

4-2 Emergence of non-Abelian potentials

To show how a Hamiltonian of the type (V.54) can arise, we will once again take a situation characteristic of quantum optics, with an atom having $N +$

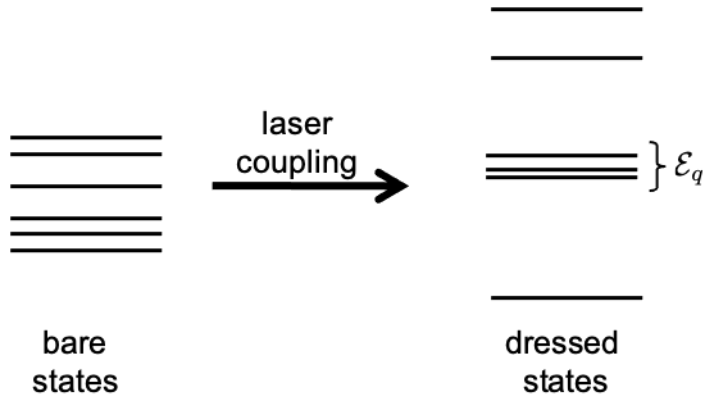


Figure V.9. When a degenerate, or nearly degenerate, subspace $\mathcal{E}_q$ appears in the spectrum of the dressed atom, restricting the dynamics to this subspace can generate non-Abelian potentials.

1 internal levels coupled by a light field. At every point $\mathbf{r}$ in space, we assume that the Hamiltonian describing the internal dynamics of the atom can be written as a time-independent matrix $H_{\text{interne}}(\mathbf{r})$ of size $(N+1) \times (N+1)$. For a given position $\mathbf{r}$, one can diagonalize the matrix $H_{\text{interne}}(\mathbf{r})$ and obtain $N+1$ independent dressed states, which we will denote $|\psi_n(\mathbf{r})\rangle$, with their energy $E_n(\mathbf{r})$ ($n = 1, \dots, N+1$). Our starting hypothesis will be that, among these $N+1$ dressed states, a number q of them (the first q , for example) form an (almost) degenerate subspace $\mathcal{E}_q$, and that these levels are well separated from the others (figure V.9).

The quantum state of the atom, including both its internal degrees of freedom and its center of mass, can be written

$$\Psi(\mathbf{r}) = \sum_{n=1}^{N+1} \phi_n(\mathbf{r}) |\psi_n(\mathbf{r})\rangle \quad (\text{V.65})$$

where each ϕ_n is the wave function describing the motion of the center of mass for the atom in the internal state $|\psi_n\rangle$. We are interested in the dynamics of the atom when it is initially prepared in the subspace $\mathcal{E}_q$. Neglecting transitions outside this subspace, we can project the complete Schrödinger

equation onto $\mathcal{E}_q$ and arrive, for the column vector

$$\Phi(\mathbf{r}) = \begin{pmatrix} \phi_1(\mathbf{r}) \\ \vdots \\ \phi_q(\mathbf{r}) \end{pmatrix} \quad (\text{V.66})$$

at a closed equation that takes the structure of a Schrödinger equation

$$i\hbar \frac{\partial \Phi}{\partial t} = \left[\frac{(\hat{\mathbf{p}} - \hat{\mathcal{A}}(\hat{\mathbf{r}}))^2}{2M} + \hat{E}(\hat{\mathbf{r}}) + \hat{\mathcal{V}}(\hat{\mathbf{r}}) \right] \Phi, \quad (\text{V.67})$$

where $\hat{E}(\mathbf{r})$ is the $q \times q$ diagonal matrix formed from the eigenenergies $E_n(\mathbf{r})$ ($n = 1, \dots, q$).

This equation strongly recalls the Schrödinger equation found in the scalar case, Eq. (V.12). However, the Berry connection $\hat{\mathcal{A}}$ and the additional potential $\hat{\mathcal{V}}$ are now $q \times q$ matrices, whose elements are

$$\hat{\mathcal{A}}^{(n,m)} = i\hbar \langle \psi_n(\mathbf{r}) | \nabla \psi_m(\mathbf{r}) \rangle, \quad (\text{V.68})$$

$$\hat{\mathcal{V}}^{(n,m)} = \frac{1}{2M} \sum_{l=q+1}^{N+1} \hat{\mathcal{A}}^{(n,l)} \cdot \hat{\mathcal{A}}^{(l,m)}, \quad (\text{V.69})$$

with $n, m \in (1, \dots, q)$. In the case where one restricts oneself to a one-dimensional subspace $\mathcal{E}_q$, one checks that one recovers the result obtained for the Abelian case.

The Berry curvature associated with $\mathcal{A}$ is likewise a $q \times q$ matrix:

$$\mathcal{B}_i = \frac{1}{2} \epsilon_{ikl} F_{kl}, \quad F_{kl} = \partial_k \mathcal{A}_l - \partial_l \mathcal{A}_k - \frac{i}{\hbar} [\mathcal{A}_k, \mathcal{A}_l]. \quad (\text{V.70})$$

As mentioned above, the artificial magnetic field $\mathcal{B}$ can be nonzero even if the vector potential $\mathcal{A}$ is uniform in space.

4-3 The multipod level scheme

A generic way of obtaining a degenerate subspace in an atom-laser interaction is to start from the situation shown in figure V.10, where a single

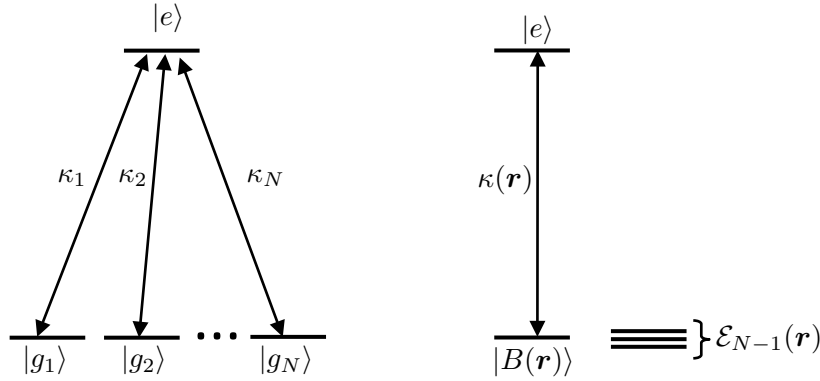


Figure V.10. Multipod configuration: an excited atomic state $|e\rangle$ is coupled to N different atomic states $|g_j\rangle$ ($j = 1, \dots, N$) by N resonant laser fields. This coupling can be rewritten by introducing a bright state $|B(\mathbf{r})\rangle$ and a dark subspace of dimension $N - 1$.

excited state $|e\rangle$ is coupled to N ground levels $|g_j\rangle$ ($j = 1, \dots, N$) [figure V.10]. Each coupling $|g_j\rangle \leftrightarrow |e\rangle$ is provided by a laser beam of angular frequency ω_j and complex Rabi frequency κ_j . This level scheme occurs, for example, when one considers an atomic transition between a ground level of angular momentum $J_g = 1$ and an excited level of angular momentum $J_e = 0$, corresponding to $N = 3$. By considering Raman transitions between hyperfine states, one can also explicitly construct schemes making it possible to realize $N = 4$ for alkali atoms (Juzeliūnas, Ruseckas, et al. 2010).

We will assume here that the Raman couplings between the various sublevels are all resonant, i.e., that

$$E_{g_j} + \hbar\omega_j = E_e + \hbar\Delta, \quad (\text{V.71})$$

where Δ is independent of j . The Hamiltonian describing the internal dynamics of the atom is then written, within the rotating-wave approximation:

$$H_{\text{interne}} = -\hbar\Delta|e\rangle\langle e| + \sum_{j=1}^N \left(\frac{\hbar\kappa_j(\mathbf{r})}{2} |e\rangle\langle g_j| + \text{h.c.} \right). \quad (\text{V.72})$$

This Hamiltonian can be rewritten:

$$H_{\text{interne}} = -\hbar\Delta|e\rangle\langle e| + \frac{\hbar\kappa(\mathbf{r})}{2} (|e\rangle\langle B(\mathbf{r})| + |B(\mathbf{r})\rangle\langle e|), \quad (\text{V.73})$$

where we have introduced the *bright state*

$$|B(\mathbf{r})\rangle = \frac{1}{\kappa} \sum_{j=1}^N \kappa_j^* |g_j\rangle, \quad (\text{V.74})$$

and defined the total Rabi frequency κ (real, positive) by

$$\kappa^2 = \sum_{j=1}^N |\kappa_j|^2. \quad (\text{V.75})$$

The diagonalization of $H_{\text{interne}}(\mathbf{r})$ at a given point $\mathbf{r}$ makes it possible to determine the dressed states $|\psi_n\rangle$. The expression for these states is simply computed:

- The coupling between the bright state and the excited state gives rise to two dressed states, linear combinations of $|e\rangle$ and $|B\rangle$ with amplitudes that are functions of Δ and κ , as for a two-level atom.
- The subspace of the ground level of dimension $N - 1$, orthogonal to $|B(\mathbf{r})\rangle$, forms an eigenspace of H_{interne} for the eigenvalue 0.

We are thus indeed in the situation required to see a non-Abelian gauge field appear. We have formed an eigenspace $\mathcal{E}_q$ of dimension $q = N - 1$. Since the bright state depends on the point $\mathbf{r}$, the orthogonal subspace $\mathcal{E}_q$ also depends on the position of the center of mass of the atom.

We will not develop here the general calculation of the matrix $\mathcal{A}$ giving the Berry connection for this multipod configuration, but we will illustrate it on a simple case corresponding to the spin-orbit coupling.

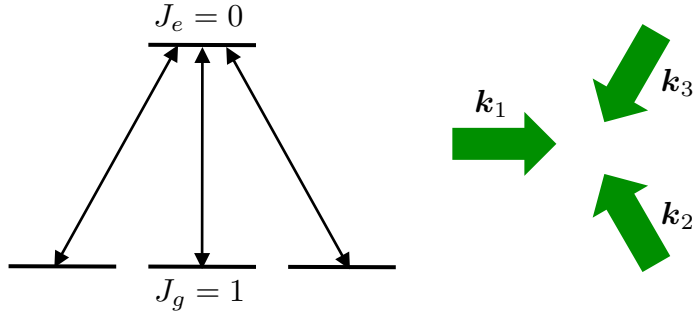


Figure V.11. Coupling of a ground state of angular momentum 1 to an excited state of zero angular momentum with three coplanar light beams (plane waves). This coupling gives rise to a two-dimensional spin-orbit effect.

5 Spin-orbit coupling

5-1 How to generate a 2D spin-orbit coupling

Let us return to the atomic configuration of the previous section, and let us focus in particular on the case of a tripod, which can be realized experimentally by coupling a ground level of angular momentum $J_g = 1$ with an excited state of angular momentum $J_e = 0$. Let us further assume that the three laser beams inducing the three couplings $|g_j\rangle \leftrightarrow |e\rangle$, $j = 1, 2, 3$, are plane waves⁴ propagating in the xy plane at angles of 120° to one another, with wave vectors:

$$\mathbf{k}_1 = k\mathbf{u}_x, \quad \mathbf{k}_2 = -\frac{k}{2}(\mathbf{u}_x - \sqrt{3}\mathbf{u}_y), \quad \mathbf{k}_3 = -\frac{k}{2}(\mathbf{u}_x + \sqrt{3}\mathbf{u}_y). \quad (\text{V.76})$$

The three complex Rabi frequencies are thus

$$\kappa_j = \frac{\kappa}{\sqrt{3}} e^{i\mathbf{k}_j \cdot \mathbf{r}}. \quad (\text{V.77})$$

⁴The three polarizations are chosen linear and mutually orthogonal, forming a frame X, Y, Z . The states $|g_j\rangle$ are then the sublevels $|m_X = 0\rangle$, $|m_Y = 0\rangle$, $|m_Z = 0\rangle$ of the level $|g, J_g = 1\rangle$.

The bright state is written, in the basis $\{|g_1\rangle, |g_2\rangle, |g_3\rangle\}$, as:

$$|B(\mathbf{r})\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} e^{-i\mathbf{k}_1 \cdot \mathbf{r}} \\ e^{-i\mathbf{k}_2 \cdot \mathbf{r}} \\ e^{-i\mathbf{k}_3 \cdot \mathbf{r}} \end{pmatrix} \quad (\text{V.78})$$

and one can explicitly write an orthogonal basis $\{|N_1\rangle, |N_2\rangle\}$ (among an infinite number of others) of the “dark” subspace, orthogonal to the bright state:

$$|N_1\rangle = \begin{pmatrix} e^{-i\mathbf{k}_1 \cdot \mathbf{r}} \\ e^{-i(\mathbf{k}_2 \cdot \mathbf{r} + 2\pi/3)} \\ e^{-i(\mathbf{k}_3 \cdot \mathbf{r} + 4\pi/3)} \end{pmatrix}, \quad |N_2\rangle = \begin{pmatrix} e^{-i\mathbf{k}_1 \cdot \mathbf{r}} \\ e^{-i(\mathbf{k}_2 \cdot \mathbf{r} - 2\pi/3)} \\ e^{-i(\mathbf{k}_3 \cdot \mathbf{r} - 4\pi/3)} \end{pmatrix}. \quad (\text{V.79})$$

One can then compute the matrix giving the Berry connection, component by component. One thus finds

$$\begin{aligned} \hat{\mathcal{A}}^{(1,1)} &= i\hbar \langle N_1 | \nabla N_1 \rangle \\ &= \frac{\hbar}{3} \left(e^{i\mathbf{k}_1 \cdot \mathbf{r}}, e^{i(\mathbf{k}_2 \cdot \mathbf{r} + 2\pi/3)}, e^{i(\mathbf{k}_3 \cdot \mathbf{r} + 4\pi/3)} \right) \cdot \begin{pmatrix} \mathbf{k}_1 e^{-i\mathbf{k}_1 \cdot \mathbf{r}} \\ \mathbf{k}_2 e^{-i(\mathbf{k}_2 \cdot \mathbf{r} + 2\pi/3)} \\ \mathbf{k}_3 e^{-i(\mathbf{k}_3 \cdot \mathbf{r} + 4\pi/3)} \end{pmatrix} \\ &= \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 \\ &= 0, \end{aligned} \quad (\text{V.80})$$

and

$$\begin{aligned} \hat{\mathcal{A}}^{(2,1)} &= i\hbar \langle N_2 | \nabla N_1 \rangle \\ &= \frac{\hbar}{3} \left(e^{i\mathbf{k}_1 \cdot \mathbf{r}}, e^{i(\mathbf{k}_2 \cdot \mathbf{r} - 2\pi/3)}, e^{i(\mathbf{k}_3 \cdot \mathbf{r} - 4\pi/3)} \right) \cdot \begin{pmatrix} \mathbf{k}_1 e^{-i\mathbf{k}_1 \cdot \mathbf{r}} \\ \mathbf{k}_2 e^{-i(\mathbf{k}_2 \cdot \mathbf{r} + 2\pi/3)} \\ \mathbf{k}_3 e^{-i(\mathbf{k}_3 \cdot \mathbf{r} + 4\pi/3)} \end{pmatrix} \\ &= \mathbf{k}_1 + \mathbf{k}_2 e^{i2\pi/3} + \mathbf{k}_3 e^{-i2\pi/3} \\ &= \frac{\hbar k}{2} (\mathbf{u}_x + i\mathbf{u}_y). \end{aligned} \quad (\text{V.81})$$

Likewise, one finds

$$\hat{\mathcal{A}}^{(1,2)} = \frac{\hbar k}{2} (\mathbf{u}_x - i\mathbf{u}_y), \quad \hat{\mathcal{A}}^{(2,2)} = 0, \quad (\text{V.82})$$

so that the *Berry connection* matrix takes a simple form using the Pauli matrices⁵:

$$\hat{\mathcal{A}} = \frac{\hbar k}{2} (\hat{\sigma}_x \mathbf{u}_x + \hat{\sigma}_y \mathbf{u}_y) = \frac{\hbar k}{2} \hat{\sigma}. \quad (\text{V.83})$$

We thus find here a simple situation where the Berry connection is uniform in space, but corresponds to a nonzero curvature. The general expression (V.70) gives, in this particular case,

$$\hat{\mathcal{B}} = \frac{\hbar k^2}{2} \hat{\sigma}_z \mathbf{u}_z, \quad (\text{V.84})$$

i.e., a field uniform in space whose orientation depends on the internal state of the atom.

5-2 The physics of spin-orbit coupling

Spin-orbit coupling is a well-known phenomenon in atomic physics and in condensed-matter physics. Its origin is generally relativistic: a charged particle moves at velocity v in a region where an electric field $\mathcal{E}$ is present. In the frame of the particle, a motional magnetic field $\mathcal{B} \propto v\mathcal{E}$ appears. The interaction between this magnetic field and the magnetic moment of the particle, itself proportional to the spin S , gives rise to spin-orbit coupling.

In atomic physics, the electric field $\mathcal{E}$ to which an electron is subject is the Coulomb field created by the nucleus, and the spin-orbit coupling can be written, after rearrangement, as $\propto \mathbf{L} \cdot \mathbf{S}$, where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the orbital angular momentum of the electron considered. In solids, the electric field appears, for example, at the interface⁶ between two materials, where a two-dimensional electron gas is realized [Rashba coupling, see for example Galitski & Spielman (2013) for more details].

The applications of spin-orbit coupling in materials physics are currently the subject of intense research around spintronics, taking advantage

⁵Let us recall their definition:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

⁶An electric field can also appear in the bulk of the material if it lacks inversion symmetry, the spin-orbit coupling then being of Dresselhaus type.

of the control one can have between the spin and the momentum of conduction electrons by means of an external electric field.

On the fundamental level, spin-orbit coupling can also lead to numerous spectacular effects, among which we can cite:

- Spin-orbit coupling can give rise to materials called *topological insulators*, which are insulating in the bulk but conducting on the surface (Hasan & Kane 2010). The surface conduction properties are topologically robust, in a manner analogous (especially for the 2D case) to the edge states we encountered for the quantum Hall effect. However, unlike the Hall effect, there is here no external magnetic field breaking time-reversal invariance. Moreover, the temperature conditions needed to reach these topological-insulator states are less stringent than those required for the quantum Hall effect (Hasan & Kane 2010).
- For a particle of spin S , the spin-orbit Hamiltonian

$$\hat{H} = \frac{(\hat{\mathbf{p}} - \eta \hat{\mathbf{S}})^2}{2M} \quad (\text{V.85})$$

has an infinitely degenerate ground level. Indeed, all momentum states $\mathbf{p}$ such that $|\mathbf{p}| = \eta S$ have zero energy if the spin orientation is correctly chosen. For bosons, this infinite degeneracy of the ground state radically changes the nature of the Bose-Einstein transition (Zhou & Cui 2013).

- Consider a contact (*s*-wave) interaction for a gas of fermions; it is known that if the gas is polarized, the result of this interaction is zero because of the Pauli principle. In the presence of spin-orbit coupling, an effective interaction appears even if the gas is constrained to occupy a single band in the energy diagram. This makes it possible to envisage the formation of $p_x + ip_y$ -type superfluids from “traditional” *s*-wave interactions (Zhang, Tewari, et al. 2008).
- For a Fermi gas in contact with a superconductor, spin-orbit coupling can give rise, in a two-dimensional sample, to states with nontrivial topology, with zero-energy Majorana modes within a vortex (Sau, Lutchyn, et al. 2010). These Majorana modes can still be present in one

dimension (Oreg, Refael, et al. 2010; Lutchyn, Sau, et al. 2010), and the recent experiment of Mourik, Zuo, et al. (2012) has provided signatures of their existence, even though these results remain debated.

5-3 The 1D version of spin-orbit coupling

The two-dimensional spin-orbit coupling we have just considered has not yet been realized with cold atoms. On the other hand, a simplified one-dimensional version, initially identified by Higbie & Stamper-Kurn (2002), has been observed by several groups, starting with the NIST group for a Bose gas (Lin, Compton, et al. 2011a), then by Cheuk, Sommer, et al. (2012) and Wang, Yu, et al. (2012) for fermions.

This 1D version is in fact implicit in the exact treatment we have already presented of the atom-light interaction in a plane wave (§ 2). The origin of the coupling between external degrees of freedom (the momentum of the atom) and internal ones (the occupied level) is simply the Doppler effect. Let us note right away a major difference from the multi-dimensional case: the coupling obtained in this case is Abelian.

We can take over as it stands the form of the Hamiltonian corresponding to the Raman transition $|g_1\rangle \leftrightarrow |g_2\rangle$ considered at the beginning of the chapter

$$\hat{H}(\mathbf{p}) = \begin{pmatrix} (\mathbf{p} - \hbar\mathbf{k})^2/2M + \hbar\Delta/2 & \hbar\kappa_0/2 \\ \hbar\kappa_0/2 & (\mathbf{p} + \hbar\mathbf{k})^2/2M - \hbar\Delta/2 \end{pmatrix}, \quad (\text{V.86})$$

where the expression is written in the basis of “families”:

$$\mathcal{F}(\mathbf{p}) = \{|g_1, \mathbf{p} - \hbar\mathbf{k}\rangle, |g_2, \mathbf{p} + \hbar\mathbf{k}\rangle\}. \quad (\text{V.87})$$

This Hamiltonian can be rewritten in terms of the Pauli matrices:

$$\hat{H}(\mathbf{p}) = \frac{1}{2M} (\mathbf{p} - \hat{\mathcal{A}})^2 + \frac{\hbar\Delta}{2} \hat{\sigma}_z + \frac{\hbar\kappa_0}{2} \hat{\sigma}_x \quad \text{with} \quad \hat{\mathcal{A}} = \hbar\mathbf{k} \hat{\sigma}_z. \quad (\text{V.88})$$

which corresponds, almost trivially, to the sought-after form! It is clear that the three components of $\hat{\mathcal{A}}$ commute with one another. Since $\hat{\mathcal{A}}$ moreover does not depend on position, the Berry curvature $\hat{\mathcal{B}}$ defined in (V.62) is zero in this particular case.

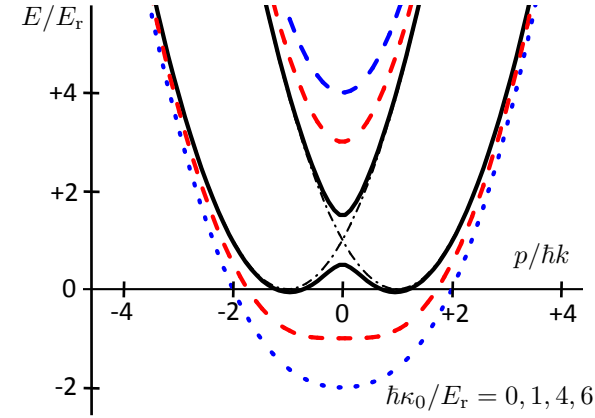


Figure V.12. Energy levels of (V.86) for a detuning $\Delta = 0$ and for increasing values of the Rabi frequency κ_0 .

Despite the passage to one dimension, certain characteristics of spin-orbit coupling remain relevant. For example, for $\Delta = 0$ and a sufficiently small amplitude of the Raman coupling, the non-uniqueness of the ground state of the single-particle Hamiltonian is preserved. We have plotted in figure V.12 the variation of the lowest energy level of (V.86)

$$E_-(p) = \frac{p^2}{2M} + E_r - \frac{\hbar}{2} \left[\kappa_0^2 + 4 \left(\frac{kp}{M} \right)^2 \right]^{1/2}. \quad (\text{V.89})$$

One finds that this energy is minimal at two symmetric, nonzero momenta p and $-p$, as long as the Rabi frequency satisfies

$$\hbar\kappa_0 < 4E_r. \quad (\text{V.90})$$

Beyond this value, the minimum of $E_-(p)$ is located at $p = 0$, and one finds oneself approximately in the situation of the adiabatic approximation studied in section 2. The case of interest from the spin-orbit point of view, $\hbar\kappa_0 < 4E_r$, is thus the opposite of the case where the adiabatic approximation is valid. Note that each of the minima is associated with a predominant internal state, the state $|g_2\rangle$ for the minimum at $p < 0$ and the state $|g_1\rangle$ for the minimum at $p > 0$. We have reported in figure V.13

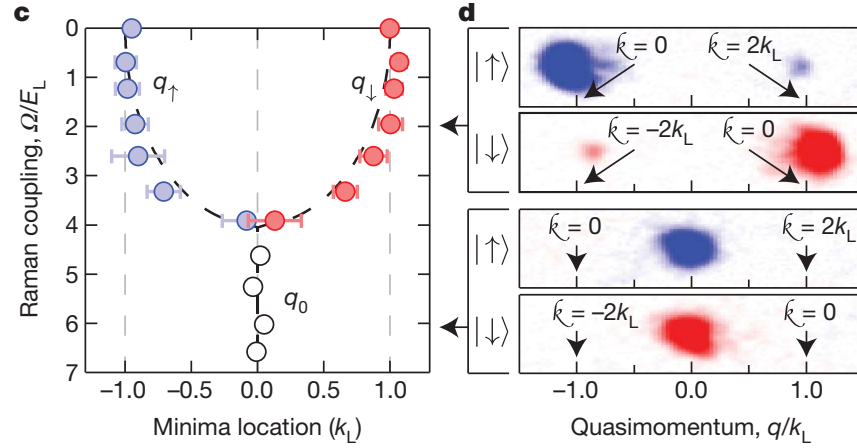


Figure V.13. Results taken from Lin, Compton, et al. (2011a) showing the evolution of the positions of the minima of $E_-(p)$ with the Rabi angular frequency κ_0 . The color codes the relative populations of the two Zeeman sublevels involved $m_F = 0$ and $m_F = -1$.

results obtained in the NIST group, which fully confirmed this simple picture for rubidium atoms, by isolating two Zeeman sublevels ($m_F = 0$ and $m_F = -1$) of the $F = 1$ level.

At the single-particle level, obtaining and describing the 1D spin-orbit coupling is thus extremely simple, and amounts more to a reinterpretation of the usual energy diagram of a dressed atom than to a new configuration. On the other hand, as soon as one seeks to take into account interactions between particles, spin-orbit coupling considerably enriches the problem; numerous questions remain open regarding the phases likely to appear both in Bose gases (see for example Li, Pitaevskii, et al. (2012)) and for fermions, with the already-mentioned possibility of generating Majorana particles at the ends of an atomic chain in a topological superconducting state (Oreg, Refael, et al. 2010; Lutchyn, Sau, et al. 2010).

Chapter VI

Magnetism in a periodic lattice

So far we have been interested in generating an artificial gauge field for an atom moving freely in space or confined in a harmonic potential. Another very important class of problems concerns orbital magnetism in the presence of a spatially periodic potential. This type of question can arise, for example, when studying the effect of a strong magnetic field on the electron fluid of a crystal.

The richness of this problem is related to the existence of two length scales that can be of the same order. The first is the lattice spacing a of the periodic lattice, which we can take to be square for definiteness, the second is the magnetic length we have already encountered

$$\ell = \sqrt{\hbar / qB}. \quad (\text{VI.1})$$

The ratio of these two lengths can be written in terms of the flux $\Phi = Ba^2$ of the magnetic field through a unit cell, expressed in units of the flux quantum $\Phi_0 = h/q$:

$$\frac{a^2}{\ell^2} = \frac{qBa^2}{\hbar} = 2\pi \frac{\Phi}{\Phi_0}. \quad (\text{VI.2})$$

Recall that this quantity can be understood as the Aharonov–Bohm phase accumulated by the wave function of a particle moving around the edge of one of the lattice cells.

As long as the flux Φ remains small compared to Φ_0 , one does not expect any dramatic changes to the physics found for a free particle: the lattice spacing is much smaller than the typical size of the cyclotron orbits, which

should therefore not be greatly affected by the discretization of space. This regime $\Phi \ll \Phi_0$ corresponds to the case of “ordinary” solids (lattice spacing a of a few angstroms) and fields achievable in the laboratory (a few tens of Teslas at most).

On the other hand, for large magnetic fields such that $\Phi \sim \Phi_0$, the “competition” between these two length scales will induce radically new physics, with a single-particle spectrum that we will see possesses a fractal structure. This regime can be explored with “synthetic” materials and/or artificial magnetic fields, in particular cold atoms confined in an optical lattice. This problem of quantum motion in the simultaneous presence of a periodic potential and a magnetic field has been the subject of several seminars associated with this lecture series. In this chapter, we will focus on a few important points, without seeking to be exhaustive on this problem, which is both rich and complex.

1 Gauge fields on a lattice

1-1 Reminders: the tight-binding model

We will begin our study with a few reminders about the simple model of a 1D or 2D lattice in the tight-binding regime, restricting ourselves to a single energy band. This problem was studied in detail in last year’s course, and

we will therefore restrict ourselves to giving the elements useful for what follows.

Consider a one-dimensional periodic lattice, with spatial period a (figure VI.1). We denote by $|w_j\rangle$ the state of the atom when it is localized on the site located at $x_j = ja$ ($j \in \mathbb{Z}$). The dynamics of the atom in the lattice is characterized by the tunneling coefficient J , which gives the probability amplitude for hopping from site j to one of the two neighboring sites $j \pm 1$. The Hamiltonian describing the motion of the atom is thus

$$\hat{H} = -J (\hat{T} + \hat{T}^\dagger) \quad (\text{VI.3})$$

where the operator $\hat{T}$ translates the particle by one site to the right

$$\hat{T} = \sum_{j \in \mathbb{Z}} |w_{j+1}\rangle \langle w_j|. \quad (\text{VI.4})$$

The eigenstates of this Hamiltonian are the Bloch functions, which can be seen as the transposition of the notion of a plane wave to this discretized space:

$$|\psi_q\rangle = \sum_j e^{ijaq} |w_j\rangle, \quad (\text{VI.5})$$

the quasi-momentum q being chosen by convention in the Brillouin zone $q \in]-\pi/a, \pi/a]$. The energy of a state $|\psi_q\rangle$ is given by

$$E(q) = -2J \cos(aq), \quad (\text{VI.6})$$

which leads to an energy band centered at 0 and of width $4J$.

The two-dimensional case is a direct generalization of the above, at least for a square lattice. A site is now labeled by two integers j, l , corresponding to the two spatial directions x, y . The (separable) Hamiltonian is written

$$\hat{H} = \hat{H}_x + \hat{H}_y \quad \text{with} \quad \hat{H}_\nu = -J (\hat{T}_\nu + \hat{T}_\nu^\dagger), \quad \nu = x, y, \quad (\text{VI.7})$$

or else, explicitly:

$$\hat{H} = -J \sum_{j,l} (|w_{j+1,l}\rangle \langle w_{j,l}| + |w_{j,l+1}\rangle \langle w_{j,l}|) + \text{h.c.} \quad (\text{VI.8})$$

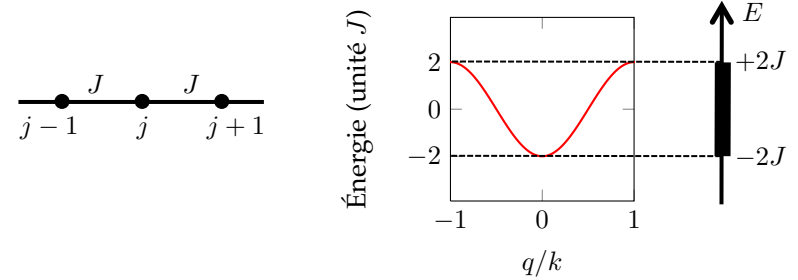


Figure VI.1. 1D lattice in the tight-binding (Hubbard) regime, with an energy band of width $4J$.

Its eigenvectors are Bloch functions $|\psi_q\rangle$, now labeled by a vector $\mathbf{q} = (q_x, q_y)$ and having energy $E(\mathbf{q})$

$$|\psi_q\rangle = \sum_{j,l} e^{ia(jq_x + lq_y)} |w_{j,l}\rangle, \quad E(\mathbf{q}) = -2J (\cos(aq_x) + \cos(aq_y)), \quad (\text{VI.9})$$

i.e., an energy band centered at 0 and of width $8J$.

1-2 Lattice under a magnetic field

In the presence of a magnetic field perpendicular to the plane of the lattice, the Hamiltonian (VI.8) must be modified to take into account the Aharonov–Bohm phase

$$\phi(\mathbf{r} \rightarrow \mathbf{r}') = \frac{q}{\hbar} \int_{\mathbf{r}}^{\mathbf{r}'} \mathbf{A} \cdot d\mathbf{r} \quad (\text{VI.10})$$

along the paths going from a site $\mathbf{r} = (j, l)$ to the four neighboring sites $\mathbf{r}' = (j \pm 1, l)$ and $(j, l \pm 1)$. An approximate way to do this is the *Peierls substitution*¹, which consists in assigning – in this tight-binding model –

¹For charges moving in a lattice in the presence of a magnetic field, the passage from the underlying Hamiltonian to this model Hamiltonian with complex tunneling coefficients is far from obvious. The limitations of this *Peierls substitution* have been discussed by several authors (Luttinger 1951; Nenciu 1991); however, in the context of the simulation of magnetism

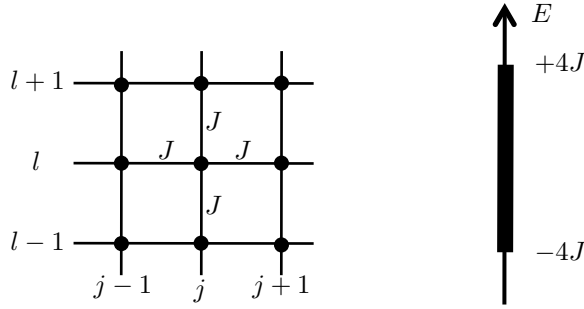


Figure VI.2. 2D square lattice in the tight-binding (Hubbard) regime, with an energy band of width $8J$.

different phases to the tunneling coefficients along the horizontal and vertical links that form the 2D square lattice:

$$-J |w_{j+1,l}\rangle \langle w_{j,l}| \longrightarrow -J e^{i\phi(j,l \rightarrow j+1,l)} |w_{j+1,l}\rangle \langle w_{j,l}| \quad (\text{VI.11})$$

with, of course,

$$\phi(j, l \rightarrow j+1, l) = -\phi(j+1, l \rightarrow j, l) \quad (\text{VI.12})$$

to guarantee that the Hamiltonian remains Hermitian.

1-3 Gauge choice

Just as there is an arbitrariness in the choice of gauge for a particle moving in the continuous xy plane, there is likewise a gauge arbitrariness in the choice of these phases. To convince ourselves of this, let us first consider the 4-site model of figure VI.3, corresponding to the Hamiltonian

$$\hat{H} = -J (e^{i\phi_1} |a\rangle \langle d| + e^{i\phi_2} |b\rangle \langle a| + e^{i\phi_3} |c\rangle \langle b| + e^{i\phi_4} |d\rangle \langle c|) + \text{h.c.} \quad (\text{VI.13})$$

with cold atoms, these limitations are not relevant, since we will seek to reproduce directly the model Hamiltonian (VI.21).

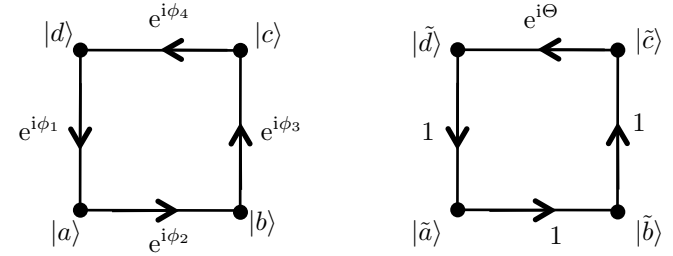


Figure VI.3. Principle of a gauge transformation, in which the states of the particle are redefined to “concentrate” the phase of the tunneling coefficients on a particular link of the cell.

Let us consider the following change of basis, which simply consists in redefining the phase of the states used:

$$|\tilde{a}\rangle = e^{i\phi_1} |a\rangle, \quad |\tilde{b}\rangle = e^{i(\phi_1+\phi_2)} |b\rangle, \quad |\tilde{c}\rangle = e^{i(\phi_1+\phi_2+\phi_3)} |c\rangle, \quad |\tilde{d}\rangle = |d\rangle. \quad (\text{VI.14})$$

One immediately sees that the Hamiltonian (VI.13) can be rewritten

$$\hat{H} = -J (|\tilde{a}\rangle \langle \tilde{d}| + |\tilde{b}\rangle \langle \tilde{a}| + |\tilde{c}\rangle \langle \tilde{b}| + e^{i\Theta} |\tilde{d}\rangle \langle \tilde{c}|) + \text{h.c.} \quad (\text{VI.15})$$

with

$$\Theta = \phi_1 + \phi_2 + \phi_3 + \phi_4. \quad (\text{VI.16})$$

In this particular case, we have thus been able to choose all the tunneling coefficients real except for one of them, which concentrates the total phase of the initial coefficients. We have placed this complex coefficient on the top link of the plaquette, but one can of course choose to place this tunneling coefficient $J e^{i\Theta}$ on any other side, the important point being that its phase equals Θ .

By generalizing this argument, one can show that the gauge-invariant quantities for an infinite square lattice are the sums

$$\begin{aligned} \frac{q}{\hbar} \Phi_{j,l} &= \phi(j, l \rightarrow j+1, l) + \phi(j+1, l \rightarrow j+1, l+1) \\ &+ \phi(j+1, l+1 \rightarrow j, l+1) + \phi(j, l+1 \rightarrow j, l). \end{aligned} \quad (\text{VI.17})$$

Each sum represents the phase accumulated along the path around a given plaquette of the lattice (figure VI.4), in this case the plaquette with the site

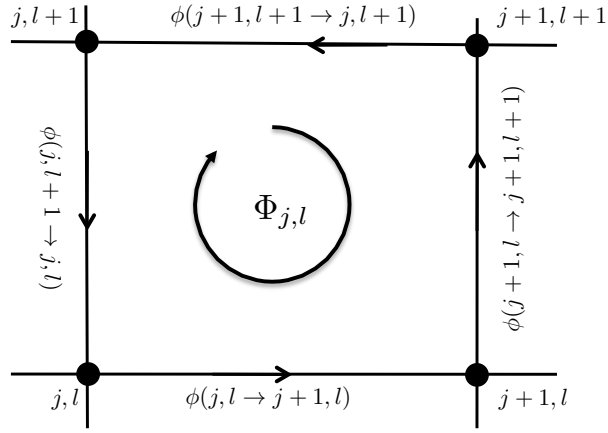


Figure VI.4. For each plaquette, the phases $\phi(j, l \rightarrow j', l')$ entering the tunneling matrix elements between nearest neighbors can be summed to form the gauge-invariant quantity $(q/\hbar) \Phi_{j,l}$, cf. (VI.17). The quantity $\Phi_{j,l}$ represents the flux of the magnetic field through this plaquette.

(j, l) at its lower left corner. All choices of $\phi(j, l \rightarrow j \pm 1, l)$, $\phi(j, l \rightarrow j, l \pm 1)$ leading to the same set $\{\Phi_{j,l}\}$ will be equivalent. The physical meaning of $\Phi_{j,l}$ is, as in the continuous case, the flux of the magnetic field through the plaquette considered.

To show how this result can be put into practice, let us adopt a strategy directly inspired by what we did for the isolated square plaquette of figure VI.3. Let us start from an infinite square lattice, on which each link corresponds to an *a priori* complex tunneling coefficient.

- Starting from the site $(j, l) = (0, 0)$ (figure VI.5a), one can begin by modifying the phase of all the sites of the horizontal line $l = 0$ (except the site $|w_{0,0}\rangle$, which serves as the starting point)

$$|w_{j,0}\rangle \rightarrow |\tilde{w}_{j,0}\rangle, \quad j \neq 0, \quad (\text{VI.18})$$

with, for example,

$$|\tilde{w}_{1,0}\rangle = e^{i\phi_{00 \rightarrow 1,0}} |w_{1,0}\rangle, \quad |\tilde{w}_{2,0}\rangle = e^{i(\phi_{00 \rightarrow 1,0} + \phi_{1,0 \rightarrow 2,0})} |w_{2,0}\rangle, \dots$$

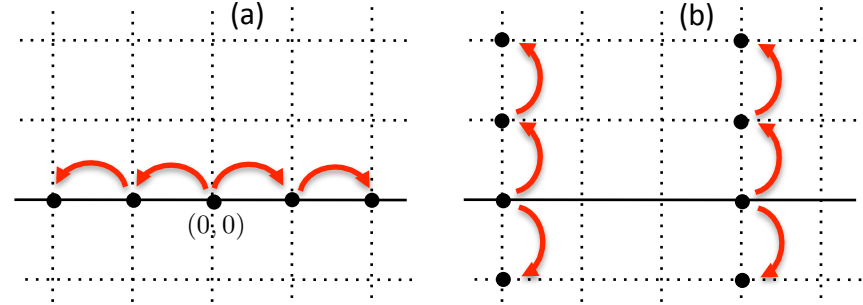


Figure VI.5. Procedure for uniquely fixing the phase of the tunneling coefficients in a square lattice. (a) The phase of the states of the horizontal line $l = 0$ is redefined so that the tunneling coefficients along this line are real. (b) The phase of the states outside the line $l = 0$ is redefined so that all the tunneling coefficients along the vertical links are real. The phases related to the presence of the magnetic field are then concentrated on the horizontal links (outside the line $l = 0$).

so that the tunneling coefficients of this horizontal line $l = 0$ become real.

- The expression for each state $|\tilde{w}_{j,0}\rangle$ being then fixed, one can consider each vertical line and redefine the phase of the coefficients of the sites along each line

$$|w_{j,l}\rangle \rightarrow |\tilde{w}_{j,l}\rangle, \quad l \neq 0, \quad (\text{VI.19})$$

so that the tunneling coefficients associated with all the vertical links are also real (figure VI.5b).

At the end of this procedure, the phase of each state $|w_{j,l}\rangle$ has been uniquely redefined; all the vertical links are real, and the phase of the horizontal links is uniquely fixed (in agreement with (VI.17)), with a zero phase for the horizontal line $l = 0$.

1-4 Landau gauge

From now on, let us consider a uniform magnetic field B , for which all the fluxes $\Phi_{j,l}$ are identical. One can make a choice similar to the Lan-

dau gauge, in which the vector potential is aligned along one direction of space, x or y . Let us choose, for example, the vector potential parallel to the x axis in (VI.10), which corresponds, in the continuous case, to $\mathbf{A} = (-By, 0, 0)$. This amounts to taking all the phases of the vertical transitions $\phi(j, l \rightarrow j, l+1)$ to be zero and choosing the phases in the horizontal direction $\phi(j, l \rightarrow j \pm 1, l)$ to be linear in l :

$$\phi(j, l \rightarrow j, l+1) = 0, \quad \phi(j, l \rightarrow j \pm 1, l) = \pm 2\pi\alpha l. \quad (\text{VI.20})$$

The Hamiltonian we will consider in what follows is thus written:

$$\hat{H} = -J \sum_{j,l} (e^{i2\pi\alpha l} |w_{j+1,l}\rangle \langle w_{j,l}| + |w_{j,l+1}\rangle \langle w_{j,l}|) + \text{h.c.} \quad (\text{VI.21})$$

The phase increment $2\pi\alpha$ entirely characterizes the problem. Since a phase is defined modulo 2π , one can restrict α to the interval $[0, 1[$ (or even $[0, 1/2]$, since α and $1 - \alpha$ represent equivalent situations under the inversion $x \rightarrow -x$). This number α is directly related to the flux Φ of the magnetic field through a plaquette:

$$\alpha = \frac{1}{2\pi} \frac{q}{\hbar} \Phi = \frac{\Phi}{\Phi_0}, \quad (\text{VI.22})$$

already introduced in (VI.2). As we noted in the introduction, for “real” crystal lattices, the lattice spacing is of order an angstrom, and the largest magnetic fields are of order 50 Teslas, which restricts α to values ($\sim 10^{-4}$) small compared to 1. Cold atoms in optical lattices constitute one way (among others) of exploring a larger part of the range of variation of α .

2 The Hofstadter butterfly

The Hamiltonian in the presence of a magnetic field (VI.21) is remarkably close to the zero-field version (VI.8). Nevertheless, the phase factor $e^{i2\pi\alpha l}$ considerably modifies the energy spectrum and the form of the eigenstates. We will not fully solve this problem, but we will give a few pointers as well as the main results.

2-1 The Harper equation

Note first that, with the Landau gauge choice we have made, we have not broken the translational invariance by a along the horizontal axis. One can thus still look for the eigenfunctions of the Hamiltonian in the form of Bloch functions² along the x axis:

$$|\Psi\rangle = \sum_{j,l} C_l e^{ijaq_x} |w_{j,l}\rangle, \quad (\text{VI.23})$$

the coefficients C_l being unknown at this stage. To determine them, we write the eigenvalue equation for $|\Psi\rangle$ and obtain the *Harper equation*

$$C_{l-1} + 2C_l \cos(2\pi\alpha l - aq_x) + C_{l+1} = -(E/J) C_l. \quad (\text{VI.24})$$

We have thus reduced the problem to the diagonalization of a tridiagonal matrix, well suited to a numerical solution.

2-2 The Hofstadter spectrum

An important class of solutions concerns the case where α is a rational number: $\alpha = p'/p$, where p and p' are two integers. One finds then that the problem also becomes invariant under translation along the vertical axis, but with an increased spatial period, pa instead of a . Indeed, in this case one has

$$\phi(j, l+p \rightarrow j+1, l+p) = 2\pi\alpha(l+p) = 2\pi\alpha l + 2\pi p' = \phi(j, l \rightarrow j+1, l) \quad (\text{VI.25})$$

where the last equality holds modulo 2π . We thus recover a problem of a particle moving in a periodic lattice, but with a unit cell of size $a \times (pa)$ and p sites per unit cell. It is then known that the fundamental band of width $8J$ will fragment into p subbands, generally separated by gaps³.

²just as we had looked for the eigenstates in the Landau gauge in the form of plane waves in one direction.

³This gap is sometimes replaced by a contact of *Dirac point* type. This is the case, for example, for $\alpha = 1/2$.

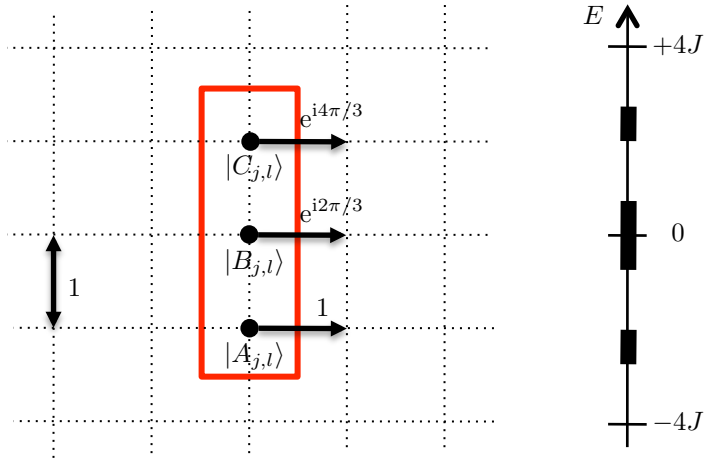


Figure VI.6. Choice of a unit cell of size $a \times 3a$ for a magnetic flux such that $\alpha = 1/3$. The three subbands resulting from the fragmentation of the initial band of width $8J$ are shown on the right.

Example of the flux $1/3$. To illustrate the general method, let us consider the example of the flux $\alpha = 1/3$. The unit cell, which has size a along the x direction and $3a$ along the y direction, contains 3 sites, denoted $|A\rangle, |B\rangle, |C\rangle$ in figure VI.6. Each cell is labeled by the indices j, l , with, by convention, site A at the point $a(j\mathbf{u}_x + 3l\mathbf{u}_y)$. Let us look for the eigenstates of the Hamiltonian in the form of a Bloch function labeled by the Bloch vector $\mathbf{q} = (q_x, q_y)$:

$$|\Psi_{\mathbf{q}}\rangle = \sum_{j,l} e^{ia(jq_x + 3lq_y)} (\alpha|A_{j,l}\rangle + \beta|B_{j,l}\rangle + \gamma|C_{j,l}\rangle) \quad (\text{VI.26})$$

with

$$q_x \in]-\pi/a, \pi/a], \quad q_y \in]-\pi/(3a), \pi/(3a)]. \quad (\text{VI.27})$$

The coefficients α, β, γ are deduced from the eigenvalue equation $\hat{H}|\Psi\rangle = E|\Psi\rangle$, which amounts to seeking the eigenstates of the Hamiltonian in reciprocal space:

$$\hat{\mathcal{H}}(\mathbf{q}) = -J \begin{pmatrix} 2\cos(aq_x) & 1 & e^{i3aq_y} \\ 1 & 2\cos(aq_x + 2\pi/3) & 1 \\ e^{-i3aq_y} & 1 & 2\cos(aq_x + 4\pi/3) \end{pmatrix} \quad (\text{VI.28})$$

written here in the basis $\{|A\rangle, |B\rangle, |C\rangle\}$. Diagonalizing this 3×3 matrix gives three eigenvalues that are functions of $\mathbf{q}$. As q_x and q_y are varied within the intervals (VI.27), 3 energy bands appear, shown in figure VI.6. The energy spectrum is symmetric about $E = 0$, and its total width is reduced compared to the value $8J$ found in the absence of a magnetic field. The lowest subband has width $\sim 0.7J$ and is separated from the intermediate subband by a gap of width $\sim 1.3J$.

General form of the spectrum. The general result for the spectrum of the Hamiltonian is plotted in figure VI.7. This spectrum exhibits a very peculiar fractal structure called the *Hofstadter butterfly* (Hofstadter 1976). One can understand the appearance of this fractal structure by comparing two very close flux values, for example $100/300$ and $101/300$, and noting that they lead to very different spectra, one with 3 subbands (figure VI.6), the other with 301 subbands. A detailed study of this fractal structure, of its self-similarity and its symmetries, as well as numerous references, can be found on the website of Gilles Montambaux⁴.

Toward weak magnetic fields: $\alpha = 1/p$. In this case one finds p subbands, and the width of the subbands tends to 0 as p increases. The lowest bands are approximately equidistant, with a spacing that also tends to 0, but more slowly than their width. The spectrum thus becomes almost discrete, which of course recalls the Landau-level structure of a free particle. This connection can be verified quantitatively, by taking for the mass of the particle the relation deduced from (VI.9) at the bottom of the band:

$$E(\mathbf{q}) \approx Ja^2 \mathbf{q}^2 \equiv \frac{\hbar^2 \mathbf{q}^2}{2M_{\text{eff}}} \Rightarrow M_{\text{eff}} = \frac{\hbar^2}{2Ja^2} \quad (\text{VI.29})$$

and by checking that the spacing between levels is indeed equal to the cyclotron angular frequency $\omega_c = qB/M_{\text{eff}}$, which is here written

$$\hbar\omega_c = 2J \frac{\Phi}{\Phi_0}. \quad (\text{VI.30})$$

⁴www.equipes.lps.u-psud.fr/Montambaux/cours-seminaires/hofstadter-tutorial.pdf

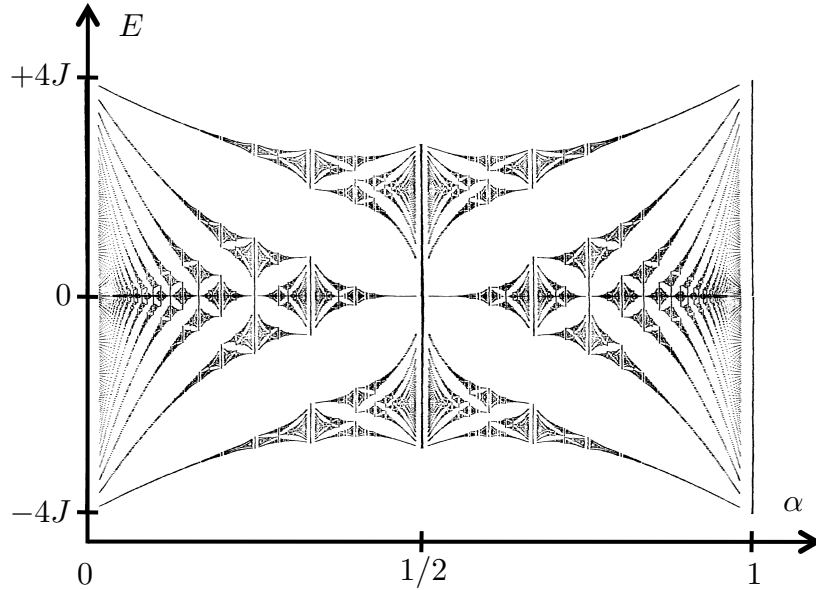


Figure VI.7. Hofstadter butterfly: energy spectrum of a particle moving on a square lattice within the tight-binding approximation, in the presence of a magnetic field. The flux Φ of the magnetic field through a plaquette of the lattice is such that $2\pi\alpha = \Phi/\Phi_0$ (Hofstadter 1976).

2-3 The Chern number

We see from the Hofstadter spectrum that Landau levels naturally appear as the asymptotic limit $\Phi \rightarrow 0$ of the subbands found for a flux $\propto 1/p$, where p is an integer. To go beyond this simple visual remark, one must quantitatively characterize the properties of these subbands to see to what extent they globally lead to the same physical properties as Landau levels for a free particle.

A remarkable property of Landau levels, independent of the precise shape of the sample, is the existence of chiral edge states, which we were able to relate to the notion of the Hall current. Let us recall the main result; take a rectangular sample and consider a situation where the lowest

Landau level is completely filled with non-interacting fermionic particles; the other Landau levels are assumed empty. If a potential difference ΔE_x is applied between the two edges of the sample along the x axis, then a particle current appears in the y direction, with value $N_y = \Delta E_x/h$. Does one obtain the same property for particles on a lattice?

The answer to this question, found by Thouless, Kohmoto, et al. (1982) [see also Kohmoto (1989)], is contained in the value of a topological invariant, the Chern number C defined by:

$$J_y = \frac{C}{h} F_x, \quad (\text{VI.31})$$

where F_x is a uniform force applied to the lattice in the x direction and J_y the particle current in the perpendicular direction y . When one assumes the lowest band filled with fermions that do not interact with one another and the other bands empty, this Chern number equals:

$$C = \frac{1}{2\pi} \int_{ZB} \Omega_z(\mathbf{q}) d q_x d q_y \quad (\text{VI.32})$$

where the integral of the function $\Omega_z(\mathbf{q})$ is taken over the first Brillouin zone. The definition of the function $\Omega_z(\mathbf{q})$ is the direct transposition to reciprocal space of the notion of Berry curvature that we introduced in position space (Xiao, Chang, et al. 2010). Since we have a periodic Hamiltonian, we can look for its eigenstates in the form of Bloch waves. Restricting ourselves to the lowest band, which by hypothesis is the only one populated, these Bloch waves are denoted

$$\Psi_{\mathbf{q}}(\mathbf{r}) = e^{i\mathbf{r} \cdot \mathbf{q}} u_{\mathbf{q}}(\mathbf{r}), \quad (\text{VI.33})$$

where $u_{\mathbf{q}}(\mathbf{r})$ is a function periodic on the lattice. One then defines the Berry connection

$$\mathcal{A}(\mathbf{q}) = i \langle u_{\mathbf{q}} | \nabla_{\mathbf{q}} u_{\mathbf{q}} \rangle, \quad (\text{VI.34})$$

which makes it possible to evaluate the geometric phase accumulated by a particle traversing a closed loop in quasi-momentum space. The Berry curvature is

$$\Omega(\mathbf{q}) = \nabla_{\mathbf{q}} \times \mathcal{A}(\mathbf{q}). \quad (\text{VI.35})$$

One can show that the Chern number thus defined is a positive, negative, or zero integer. When this integer is zero, the lattice is topologically

trivial: a force applied along x induces no macroscopic current along y . On the other hand, if it is nonzero, one is in the presence of a *Hall-current*-type phenomenon, in which a force in one direction induces a current in a perpendicular direction.

To gain an intuition for the meaning of this integer, let us return to a well-known effect of the applied force F_x : it will create a phenomenon of Bloch oscillations⁵ along the x direction, with the quasi-momentum sweeping through $\dot{q}_x = F_x/\hbar$. For a lattice with a rectangular unit cell $a \times b$, the period of the Bloch oscillation corresponds to the time to traverse the entire Brillouin zone (width $2\pi/a$), i.e., $\tau_B = \hbar/F_x a$. Let us focus on the number of particles δN that cross a unit cell, i.e., a horizontal segment of width a along the x axis, in one Bloch period τ_B :

$$\delta N = a J_y \tau_B = a \left(\frac{C}{h} F_x \right) \left(\frac{\hbar}{F_x a} \right) = C. \quad (\text{VI.36})$$

The number of particles crossing a unit cell in one Bloch period is thus quantized, and the Chern number gives the value of this quantum. For the particular rational values $1/p$ of the flux per plaquette, one finds that the Chern number of the lowest subband equals 1, just as for the lowest Landau level found in the absence of a lattice. These two systems thus have the same topological structure and lead to equivalent electrical conduction properties [see Kohmoto (1989) for a discussion of the general case $\alpha = p'/p$].

2-4 Experiments on super-lattices

As we indicated in the introduction, the lattice spacing of an ordinary crystal lattice is too small to allow the regime $\Phi/\Phi_0 \sim 1$ to be reached with realistic magnetic fields. Over the past thirty years, several classes of systems – outside of cold-atom gases – have been studied to approach the physics of the Hofstadter butterfly. Let us mention three of them here (cf. figure VI.8):

- Pannetier, Chaussy, et al. (1984) fabricated a square lattice of aluminium strips $2\mu\text{m}$ wide and spaced $6\mu\text{m}$ apart. The lattice, with

⁵Proposals for measuring the Chern number by exploiting this dynamics in cold-atom experiments have been made by Price & Cooper (2012) and Dauphin & Goldman (2013).

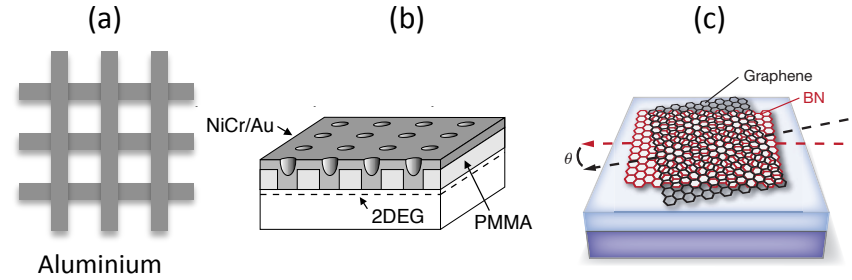


Figure VI.8. Systems used experimentally to explore the magnetic properties related to the Hofstadter butterfly (a): Pannetier, Chaussy, et al. (1984); (b): Albrecht, Smet, et al. (2001); (c): Dean, Wang, et al. (2013).

an area of 1 cm^2 , contains more than 2 million *a priori* identical sites. The measurement of the superconducting critical temperature $T_c(B)$ showed rapid variations at simple rational values ($1/2$, $2/5$, $1/3$, ...) of the flux per plaquette.

- Albrecht, Smet, et al. (2001) used a two-dimensional electron gas confined in a GaAs/AlGaAs heterostructure. A periodic lattice (period 120 nm) of NiCr/Au is deposited on the sample to spatially modulate the potential felt by the electrons. The measurement of the Hall conductance revealed the fragmentation of the spectrum into subbands, as theoretically expected.
- Dean, Wang, et al. (2013) used two sheets of graphene slightly rotated relative to one another to produce a Moiré pattern, with a periodicity of 15.5 nm . They likewise use a measurement of the Hall conductivity to explore the energy spectrum of the butterfly and to demonstrate its recursive structure.

3 Shaken optical lattices

To implement the Hamiltonian (VI.21) with a gas of cold atoms in an optical lattice, one must find a procedure for modifying the tunneling coefficient J and making it complex, with a phase that depends on the index

labeling the vertical direction l . We have already encountered a similar technique in last year's course when we were interested in dynamical localization: we had seen that the tunneling coefficient in a shaken 1D lattice could be significantly modified, or even change sign. It is this same technique that we will now review.

Note that there are different methods for simulating a magnetic field on a lattice. First, one can physically rotate the lattice in space: going into the rotating frame guarantees, as for a homogeneous gas, the appearance of an effective magnetic field (Tung, Schweikhard, et al. 2006). One can also chirally modulate the parameters of a lattice so as to simulate a rotation: for example, for a lattice made of hexagonal cells, and hence with three possible orientations of the sides, one can in principle separately modulate the three corresponding tunneling coefficients and obtain, after time-averaging, a Hamiltonian with an effective magnetic field (Kitagawa, Berg, et al. 2010) [see also the article by Sørensen, Demler, et al. (2005) for a related proposal]. Hemmerich & Smith (2007) also proposed using additional beams to generate micro-rotors at each site of the lattice; this scheme in fact realizes an alternating magnetic field, whose sign changes from one plaquette to the next [see also Lim, Smith, et al. (2008)].

3-1 Reminder: the one-dimensional case (version 1).

To prepare our analysis, let us begin by recalling how, in the one-dimensional case, one can modify the tunneling coefficient, or even give it a nonzero complex phase. Let us start with the simplest possible lattice, with the Hamiltonian

$$\hat{H}_0 = -J \sum_j (|w_{j+1}\rangle\langle w_j| + |w_j\rangle\langle w_{j+1}|) \quad (\text{VI.37})$$

Suppose that we shake this lattice by replacing the static periodic potential of the lattice $V(x)$ with

$$\mathcal{V}(x, t) = V[x - x_0(t)], \quad (\text{VI.38})$$

where $x_0(t)$ is a periodic function of time with angular frequency Ω . We will assume here that $\Omega \gg J/\hbar$, so that the modulation is fast compared to the tunneling dynamics of the lattice. We showed in last year's course that

shaking the lattice amounts, via the unitary transformation generated by the operator

$$\hat{U}(t) = \exp(i x_0(t) \hat{p} / \hbar), \quad (\text{VI.39})$$

to going over to the Hamiltonian with a complex tunneling matrix element:

$$\hat{H}(t) = -J e^{i M a \dot{x}_0(t) / \hbar} \sum_j |w_{j+1}\rangle\langle w_j| + \text{h.c.} \quad (\text{VI.40})$$

Let us write the state vector in the form

$$|\Psi(t)\rangle = \sum_j \alpha_j(t) |w_j\rangle \quad (\text{VI.41})$$

where the evolution of the amplitudes α_j is given by the Schrödinger equation

$$i \hbar \dot{\alpha}_j = -J \left(\alpha_{j+1} e^{-i M a \dot{x}_0(t) / \hbar} + \alpha_{j-1} e^{i M a \dot{x}_0(t) / \hbar} \right). \quad (\text{VI.42})$$

We will now take advantage of the fact that the α_j are slow variables in order to average the coefficients of this equation over an oscillation period⁶. We thus obtain the approximate evolution equation for the α_j

$$i \hbar \dot{\alpha}_j = -\bar{J}^* \alpha_{j+1} - \bar{J} \alpha_{j-1} \quad (\text{VI.43})$$

with a renormalized tunneling coefficient

$$\bar{J} = J \langle e^{i M a \dot{x}_0(t) / \hbar} \rangle. \quad (\text{VI.44})$$

Sinusoidal modulation. For a sinusoidal modulation of the lattice position of the type

$$\frac{M a}{\hbar} \dot{x}_0(t) = \xi_0 \sin(\Omega t + \phi) \quad (\text{VI.45})$$

where ξ_0 is a dimensionless number characterizing the amplitude of the modulation, one finds that the average tunneling coefficient is real and equal to⁷

$$\bar{J} = J \langle e^{i \xi_0 \sin(\Omega t + \phi)} \rangle = J \mathcal{J}_0(\xi_0) \quad (\text{VI.47})$$

⁶A more rigorous approach, but giving the same final result, uses the Floquet approach (cf. 2012-13 course).

⁷Recall that

$$e^{i x \sin \theta} = \sum_{n \in \mathbb{Z}} J_n(x) e^{i n \theta}. \quad (\text{VI.46})$$

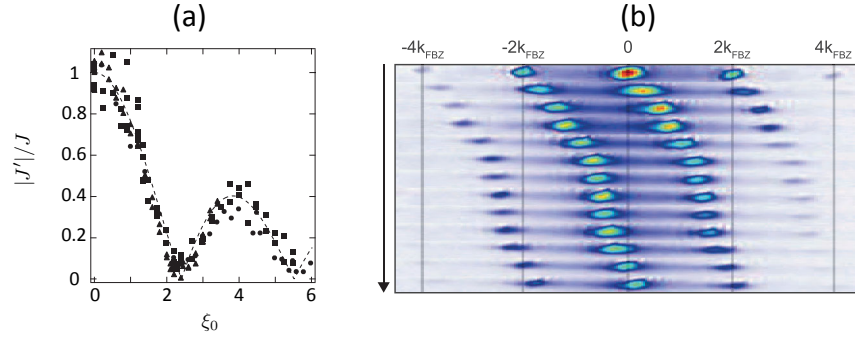


Figure VI.9. Examples of modification of the tunneling effect in one-dimensional lattices. (a) Sign change of the tunneling coefficient by a sinusoidal modulation of the lattice (Lignier, Sias, et al. 2007). (b) Phase change of the tunneling coefficient and shift of the momentum peaks observed after time of flight (Struck, Ölschläger, et al. 2012)).

where the J_n denote the Bessel functions of the first kind. This renormalization of the tunneling coefficient, with its possible sign change, was observed in an experiment carried out with rubidium atoms by Lignier, Sias, et al. (2007) (figure VI.9a).

Asymmetric modulation. Suppose now that we take a modulation of the lattice position that is not invariant under time reversal, such as the asymmetric sawtooth of figure VI.10: the lattice moves at velocity $v_1 > 0$ for a duration T_1 , then at velocity $v_2 < 0$ for a duration T_2 , with

$$v_1 T_1 + v_2 T_2 = 0, \quad T = T_1 + T_2, \quad (\text{VI.48})$$

so that the average displacement over a period T is zero. In this case, the time average of the function $e^{i M a \dot{x}_0(t)/\hbar}$ is *a priori* complex:

$$\frac{\bar{J}}{J} = \langle e^{i M a \dot{x}_0(t)/\hbar} \rangle = \frac{T_1}{T} e^{i M a v_1/\hbar} + \frac{T_2}{T} e^{i M a v_2/\hbar}. \quad (\text{VI.49})$$

In the 1D case, a nonzero phase of the tunneling coefficient manifests itself as a shift in the momentum corresponding to the bottom of the band con-

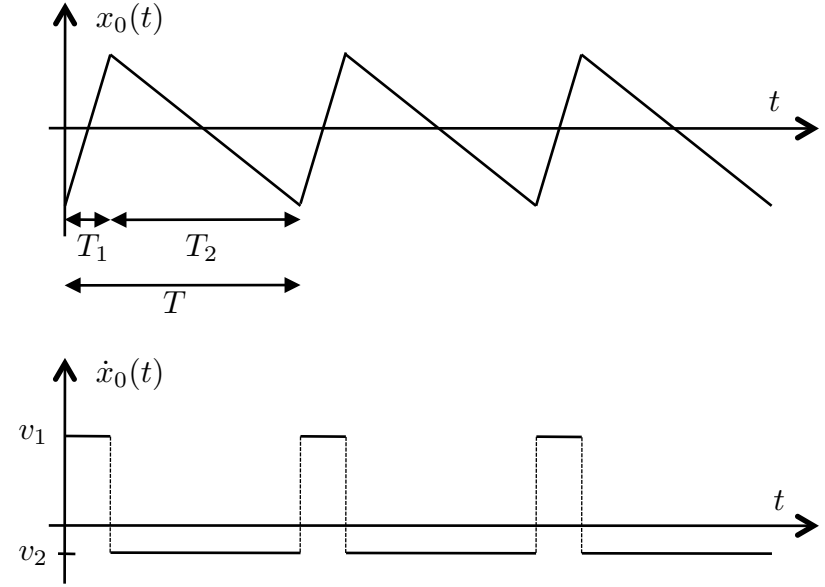


Figure VI.10. Modulation of the position of a 1D optical lattice leading to a complex tunneling coefficient (see also Struck, Ölschläger, et al. (2012)).

sidered. This shift, analogous to the one created by a uniform vector potential on a free particle and described in chapter 5, was observed by Struck, Ölschläger, et al. (2012) in a time-of-flight experiment (figure VI.9b).

3-2 The one-dimensional case (version 2).

Suppose now that the Hamiltonian $\hat{H}_0$ contains, in addition to the tunneling terms, a static linear potential

$$\hat{H}_0 = -J \sum_j (|w_{j+1}\rangle \langle w_j| + |w_j\rangle \langle w_{j+1}|) - \hbar \Omega_0 \sum_j j |w_j\rangle \langle w_j|. \quad (\text{VI.50})$$

In the absence of modulation, it is known that the eigenstates are the localized Wannier–Stark functions. Suppose now that we modulate the lattice

sinusoidally with an angular frequency Ω close to (or equal to) Ω_0 . We expect this to induce a resonance between adjacent sites that will restore conduction on the lattice. To confirm this intuition, let us proceed as above; we write the state vector at time t in the form

$$|\Psi(t)\rangle = \sum_j \alpha_j(t) e^{ij\Omega_0 t} |w_j\rangle, \quad (\text{VI.51})$$

and let us deduce, from the Schrödinger equation obtained after the unitary transformation (VI.39), the evolution of the probability amplitudes α_j

$$i\hbar \dot{\alpha}_j = -J \left(\alpha_{j+1} e^{-i(Ma\dot{x}_0(t)/\hbar - \Omega_0 t)} + \alpha_{j-1} e^{+i(Ma\dot{x}_0(t)/\hbar - \Omega_0 t)} \right). \quad (\text{VI.52})$$

For

$$\Omega = \Omega_0, \quad \frac{Ma}{\hbar} \dot{x}_0(t) = \xi_0 \sin(\Omega t + \phi), \quad (\text{VI.53})$$

one again obtains, after time-averaging, an evolution equation for the α_j that is identical to that of a static lattice:

$$i\hbar \dot{\alpha}_j = -\bar{J}^* \alpha_{j+1} - \bar{J} \alpha_{j-1} \quad (\text{VI.54})$$

with a complex renormalized tunneling coefficient:

$$\bar{J} = J \langle e^{i[\xi_0 \sin(\Omega t + \phi) - \Omega_0 t]} \rangle = J \mathcal{J}_1(\xi_0) e^{i\phi}. \quad (\text{VI.55})$$

Through this procedure we have thus succeeded in “imprinting” the phase ϕ of the modulation onto the tunneling coefficient $\bar{J}$, which was the goal sought: if this method can be generalized to 2D, we will have a means of simulating a nonzero magnetic field on the lattice.

3-3 The two-dimensional case.

We have seen above two ways of making the tunneling coefficient complex in a one-dimensional problem. Both of them can be generalized to two dimensions, but lead to different results.

- The asymmetric-modulation method of figure VI.10 can be generalized, for example, to a triangular lattice⁸ (Struck, Weinberg, et al.

⁸It does not work for a square or hexagonal lattice, or more generally for a lattice in which each plaquette has pairwise parallel edges, since the phases of the two corresponding tunneling coefficients then systematically cancel.

2013). In practice, one chooses a modulation that makes each site of the lattice rotate on a closed orbit. This method then leads to an alternating flux, with, for example, a positive phase for the “point-up” triangles and a negative phase for the “point-down” triangles.

- The method seen in § 3-2, combining a tilted lattice and resonant modulation, presents certain difficulties in being transposed as it stands [see for example the proposal of Kolovsky (2011) and its critique by Creffield & Sols (2013)]. To get around these difficulties, it is better to abandon the idea of a global modulation of the lattice, in which all the sites move by the same amount $\mathbf{r}_0(t)$, and instead move to a local modulation, in which each site j, l moves differently from its neighbors. It is this type of modulation that we will now explore (Aidelsburger, Atala, et al. 2013; Miyake, Siviloglou, et al. 2013).

Let us start from a lattice “tilted” in the horizontal direction, i.e., the base lattice to which is added a constant linear potential that shifts the energy of the site (j, l) by the amount $j\hbar\Omega_0$:

$$\hat{H}_0 = \sum_{j,l} j\hbar\Omega_0 |w_{j,l}\rangle \langle w_{j,l}| - J \sum_{j,l} (|w_{j+1,l}\rangle \langle w_{j,l}| + |w_{j,l+1}\rangle \langle w_{j,l}|) + \text{h.c.} \quad (\text{VI.56})$$

We will assume in what follows that $\hbar\Omega_0 \gg J$, which means that tunneling along the x direction is practically inhibited (the Wannier–Stark functions coincide with the Wannier functions), while tunneling along y is unchanged.

Let us add to this static term a time-dependent potential, which modulates the energy of each site with an angular frequency Ω :

$$\hat{V}(t) = \sum_{j,l} \hbar\Omega_1 \sin(\Omega t - \theta_{j,l}). \quad (\text{VI.57})$$

We will use this modulation to restore resonant tunneling ($\Omega \sim \Omega_0$) in the x direction, but the corresponding tunneling coefficient will acquire a phase that will be related to $\theta_{j,l}$. Note that such a potential can be realized by superimposing on the base optical lattice the dipole potential created by a standing wave itself composed of two traveling light waves of angular frequency ω_1 and $\omega_2 = \omega_1 + \Omega$ and wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$. The phase $\theta_{j,l}$

is then linear in j and l :

$$\theta_{j,l} = (\mathbf{k}_2 - \mathbf{k}_1) \cdot \mathbf{r} \equiv \eta_x j + \eta_y l. \quad (\text{VI.58})$$

Let us now consider the evolution of the state vector

$$|\Psi(t)\rangle = \sum_{j,l} \alpha_{j,l}(t) |w_{j,l}\rangle \quad (\text{VI.59})$$

under the effect of the total Hamiltonian $\hat{H}_0 + \hat{V}(t)$:

$$i\dot{\alpha}_{j,l} = [j\Omega_0 + \Omega_1 \sin(\Omega t - \theta_{j,l})] \alpha_{j,l} - \frac{J}{\hbar} (\alpha_{j+1,l} + \alpha_{j-1,l} + \alpha_{j,l+1} + \alpha_{j,l-1}) \quad (\text{VI.60})$$

This equation contains rapidly evolving terms, with characteristic frequencies Ω and Ω_0 , and slower terms with characteristic frequency $J/\hbar$. To work with slow variables, it is convenient to introduce

$$\tilde{\alpha}_{j,l}(t) = \alpha_{j,l}(t) \exp \left\{ i \left[j\Omega_0 t - \frac{\Omega_1}{\Omega} \cos(\Omega t - \theta_{j,l}) \right] \right\}, \quad (\text{VI.61})$$

which leads to the equation for the tilded variables

$$i\dot{\tilde{\alpha}}_{j,l} = -\frac{J}{\hbar} \left[F(j,l,t) \tilde{\alpha}_{j+1,l} + G(j,l,t) \tilde{\alpha}_{j,l+1} + \dots \right] \quad (\text{VI.62})$$

where the ellipsis corresponds to the transitions $(j,l) \rightarrow (j-1,l)$ and $(j,l) \rightarrow (j,l-1)$.

Since we are now interested in slow variables, one can average the coefficients $F(j,l,t)$ and $G(j,l,t)$ over one period of the forced oscillation $2\pi/\Omega$, placing oneself exactly at the resonance condition $\Omega = \Omega_0$. One finds:

$$\begin{aligned} \langle F(j,l,t) \rangle &= \langle \exp \left\{ i \left[-\Omega_0 t + \frac{\Omega_1}{\Omega} (\cos(\Omega t - \theta_{j+1,l}) - \cos(\Omega t - \theta_{j,l})) \right] \right\} \rangle \\ &= \langle e^{-i\Omega_0 t} \exp \left[2i \frac{\Omega_1}{\Omega} \sin((\theta_{j+1,l} - \theta_{j,l})/2) \sin(\Omega t - (\theta_{j,l} + \theta_{j+1,l})/2) \right] \rangle \\ &= \mathcal{J}_1 \left(\frac{2\Omega_1}{\Omega} \sin(\eta_x/2) \right) e^{-i\theta_{j,l}} \end{aligned} \quad (\text{VI.63})$$

where we have used $\theta_{j+1,l} - \theta_{j,l} = \eta_x$ and $(\theta_{j,l} + \theta_{j+1,l})/2 = \theta_{j,l} + \eta_x/2$ (the overall phase $e^{i\eta_x/2}$ is subsequently omitted). Likewise, one obtains

$$\langle G(j,l,t) \rangle = \mathcal{J}_0 \left(\frac{2\Omega_1}{\Omega} \sin(\eta_y/2) \right) \quad (\text{VI.64})$$

so that the effective Hamiltonian after averaging over one time period is written:

$$\hat{H} = -J_x \sum_{j,l} (e^{-i\theta_{j,l}} |w_{j+1,l}\rangle \langle w_{j,l}| + \text{h.c.}) - J_y \sum_{j,l} (|w_{j,l+1}\rangle \langle w_{j,l}| + \text{h.c.}) \quad (\text{VI.65})$$

with

$$J_x = J \mathcal{J}_1 \left(\frac{2\Omega_1}{\Omega} \sin(\eta_x/2) \right), \quad J_y = J \mathcal{J}_0 \left(\frac{2\Omega_1}{\Omega} \sin(\eta_y/2) \right). \quad (\text{VI.66})$$

To check that this Hamiltonian indeed corresponds to what we are looking for, we finally compute the flux on a plaquette:

$$\begin{aligned} \frac{q}{\hbar} \Phi_{j,l} &= \phi(j,l \rightarrow j+1,l) + \phi(j+1,l \rightarrow j+1,l+1) \\ &+ \phi(j+1,l+1 \rightarrow j,l+1) + \phi(j,l+1 \rightarrow j,l) \\ &= -\theta_{j,l} + \theta_{j,l+1} \\ &= \eta_y. \end{aligned} \quad (\text{VI.67})$$

We thus find that one must take both $\eta_y \neq 0$ for this flux to itself be nonzero, and $\eta_x \neq 0$ for the tunneling coefficient J_x to be nonzero.

3-4 Example of experiments with a shaken lattice

Several experiments have recently demonstrated the validity of the principle allowing an artificial field to be generated from a shaken lattice (Aidelsburger, Atala, et al. 2013; Miyake, Siviloglou, et al. 2013; Struck, Weinberg, et al. 2013). We show in figure VI.11 the results of the Munich group, obtained by isolating, with the help of a super-lattice, plaquettes of 2×2 sites in the xy plane. The presence of the artificial magnetic field is revealed by the cyclotron motion of the atoms in these plaquettes, with the expected chirality. These experiments were subsequently extended to the case where the plaquettes are connected to one another along one of the two directions of space, thereby realizing a ladder. In this ladder geometry, the Munich researchers demonstrated an analogue of the Meissner effect (Atala, Aidelsburger, et al. 2014).

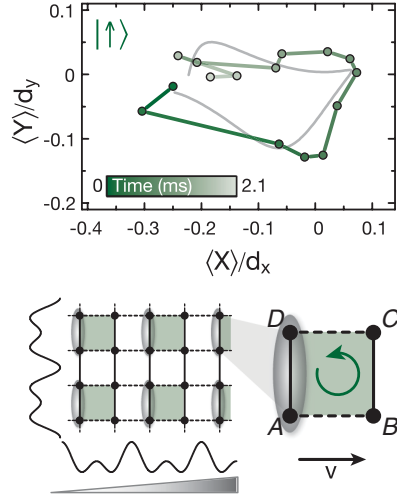


Figure VI.11. Cyclotron trajectories obtained in a shaken lattice by Aidelsburger, Atala, et al. (2013). The gray line represents a fit of the theoretically expected result, obtained from solving the Schrödinger equation for a plaquette. Each plaquette comprises 2×2 sites, and is obtained using two super-lattices in the x and y directions.

4 Lattices combining different internal states

Using several internal states in an optical lattice provides a very valuable degree of freedom for “imprinting” a phase as one moves from one site of the lattice to another. The process at play in this case is laser-assisted tunneling, which we will first present on a simple, one-dimensional system, before moving on to the case of an infinite two-dimensional lattice.

4-1 Laser-assisted tunneling on a ladder

The idea is to simulate a geometry of sites located on a two-leg ladder, shown in figure VI.12. This ladder is placed in a magnetic field, and we assume that it is well described within the tight-binding approximation.

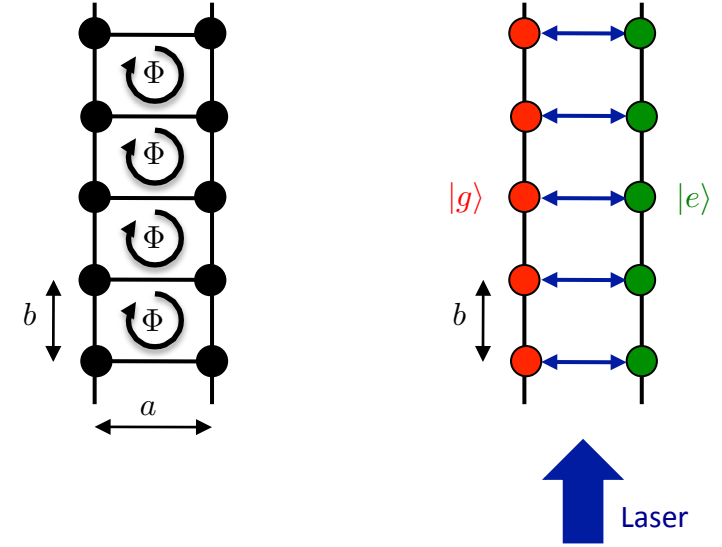


Figure VI.12. Left: infinite ladder of sites composed of two legs separated by a distance a . The rungs of the ladder are equidistant (spacing b). A charged particle can hop from one site to another by tunneling. A uniform magnetic field perpendicular to the ladder induces a flux Φ per plaquette. Right: simulation of this ladder with a double optical lattice, trapping an atom in one of the two internal states $|g\rangle$ and $|e\rangle$. A laser beam propagating along the direction of the lattice induces a transition $|g\rangle \leftrightarrow |e\rangle$, analogous to tunneling along the rungs of the ladder on the left. The laser phase $\phi = ky$ varies linearly with the index j of the rungs and “imprints” itself on the corresponding matrix element, which makes it possible to simulate a magnetic flux.

The magnetic field is, as before, characterized by the flux Φ through each plaquette.

For this simulation, we will use an atom with two internal states, denoted $|g\rangle$ and $|e\rangle$. A trapping potential, of optical-lattice type along the y direction, makes it possible to localize the atoms on the legs of the ladder. This potential is chosen to be different for $|g\rangle$ and for $|e\rangle$, which makes it possible to localize each of these two states on a different leg ($|g\rangle$ on the left, $|e\rangle$ on the right).

We assume these two internal states to be stable (no spontaneous emission), and we consider a laser transition that makes it possible to induce a resonant transition between them. In practice, if one considers atoms “with two external electrons,” such as Ytterbium, these two states will correspond respectively to the ground state (1S_0) and to the excited state of very long lifetime (3P_0). For alkali atoms, one will use for $|g\rangle$ and $|e\rangle$ two sublevels arising from the ground state, and the transition coupling these two states will be a Raman transition, involving an absorption process and a stimulated-emission process.

Let us work within a single-band model and denote by $|w_j^g\rangle$ and $|w_j^e\rangle$, $j \in \mathbb{Z}$, the spatial states for the two internal levels. For simplicity, we will assume the corresponding wave functions to be identical, up to a translation:

$$w_j^g(\mathbf{r}) = w_0(x, y - jb), \quad w_j^e(\mathbf{r}) = w_0(x - a, y - jb). \quad (\text{VI.68})$$

In the absence of the laser, one thus has two decoupled 1D optical lattices, with the Hamiltonian written in the tight-binding approximation

$$\hat{H} = \hat{H}_g + \hat{H}_e, \quad \hat{H}_\alpha = -J_y \sum_{j \in \mathbb{Z}} |w_{j+1}^\alpha\rangle \langle w_j^\alpha| + \text{h.c.}, \quad \alpha = g, e. \quad (\text{VI.69})$$

To couple these two lattices, let us consider a laser beam resonant with the transition $|g\rangle \leftrightarrow |e\rangle$, with a wave vector $\mathbf{k}$ parallel to the y axis of the lattices. The atom-laser coupling is written

$$\hat{V} = \frac{\hbar\kappa}{2} e^{iky} |e\rangle \langle g| + \text{h.c.} \quad (\text{VI.70})$$

Let us rewrite this coupling in terms of the $|w_j^\alpha\rangle$; given the restriction to a single band, we have the closure relation for the internal and external

Hilbert spaces

$$\hat{1} = |g\rangle \langle g| \otimes \left(\sum_{j \in \mathbb{Z}} |w_j^g\rangle \langle w_j^g| \right) + |e\rangle \langle e| \otimes \left(\sum_{j \in \mathbb{Z}} |w_j^e\rangle \langle w_j^e| \right). \quad (\text{VI.71})$$

By inserting this closure relation on the left and right of $\hat{V}$, one obtains

$$\hat{V} = \sum_{j, j'} \mathcal{V}_{j, j'} |e, w_j\rangle \langle g, w_{j'}| + \text{h.c.} \quad (\text{VI.72})$$

with

$$\mathcal{V}_{j, j'} = \frac{\hbar\kappa}{2} \int w_j^e(\mathbf{r}) e^{iky} w_{j'}^g(\mathbf{r}) d^2r. \quad (\text{VI.73})$$

Since we have placed ourselves within the tight-binding approximation, the wave functions $w_j^e(\mathbf{r})$ and $w_{j'}^g(\mathbf{r})$ are well localized, and one can consider their overlap to be negligible as soon as $j \neq j'$. The coefficient $\mathcal{V}_{j, j'}$ can then be simplified

$$\begin{aligned} \mathcal{V}_{j, j'} &\approx \delta_{j, j'} \frac{\hbar\kappa}{2} \int w_0(x - a, y - jb) e^{iky} w_0(x, y - jb) d^2r \\ &\approx \delta_{j, j'} J_x e^{ijkb}, \end{aligned} \quad (\text{VI.74})$$

where the tunneling coefficient J_x is proportional to the Rabi frequency and to the overlap of the Wannier functions for the internal states $|g\rangle$ and $|e\rangle$:

$$J_x = \frac{\hbar\kappa}{2} \int w_0(x - a, y) e^{iky} w_0(x, y) d^2r. \quad (\text{VI.75})$$

For a function $w_0(x, y)$ invariant under the reflection $y \rightarrow -y$, this coefficient J_x is real:

$$J_x = \frac{\hbar\kappa}{2} \int w_0(x - a, y) \cos(ky) w_0(x, y) d^2r. \quad (\text{VI.76})$$

The expression (VI.74) corresponds to what we are looking for: we have a complex tunneling coefficient whose phase grows linearly with the index j , so that the total phase accumulated around the perimeter of a plaquette is nonzero:

$$|g, j\rangle \xrightarrow{J_y} |g, j+1\rangle \xrightarrow{J_x e^{i(j+1)kb}} |e, j+1\rangle \xrightarrow{J_y} |e, j\rangle \xrightarrow{J_x e^{-ijkb}} |g, j\rangle \quad (\text{VI.77})$$

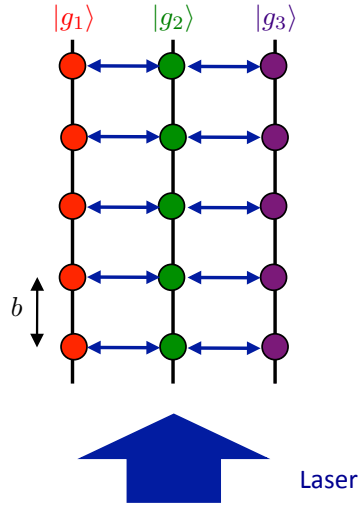


Figure VI.13. Simulation of a double ladder with a three-internal-level atom (Celi, Massignan, et al. 2014). This level scheme can give rise to a spectrum close to the Hofstadter butterfly and to edge states corresponding to opposite currents in the states $|g_1\rangle$ and $|g_3\rangle$.

i.e., a phase

$$(j+1)kb - jkb = kb \quad (\text{VI.78})$$

which is the same for all the plaquettes: we thus simulate a two-site ladder with a uniform magnetic field all along the ladder.

Synthetic-dimension lattice. One can note that in the ladder simulation above, nothing requires the length a to be nonzero. The ladder can have a dual nature, real in space along the y direction and fictitious (involving only the internal degrees of freedom) along the x direction. The need for a nonzero length a will only arise later, when one seeks to increase the number of sites in the x direction. Building on this remark, Celi, Massignan, et al. (2014) proposed extending the preceding treatment to an atom with N internal states, $|g_n\rangle$, $n = 1, \dots, N$, by choosing an atom-light coupling

that induces the transitions

$$|g_n, j\rangle \xrightarrow{J_x e^{i j k b}} |g_{n+1}, j\rangle. \quad (\text{VI.79})$$

This makes it possible to simulate a ladder whose rungs are located in real space and whose N legs are associated with the N internal states. For a practical implementation of this scheme, one can take, for example, $N = 3$ and take for the $|g_n\rangle$ the three Zeeman states $m = 0, \pm 1$ of an atomic ground level of angular momentum 1 (figure VI.13).

4-2 Extension to a two-dimensional lattice

We will now seek to extend the ladder scheme considered above to a two-dimensional lattice. In this case, the length a is genuinely real, and we consider a series of 1D lattices corresponding alternately to the internal state $|g\rangle$ and to the internal state $|e\rangle$. In this way, a two-dimensional lattice is reconstructed whose sites (j, l) are, by convention, occupied by an atom in the state $|g\rangle$ (resp. $|e\rangle$) if the index j of the site is even (resp. odd).

By precisely analyzing the phases corresponding to laser-assisted tunneling, one finds that this lattice actually corresponds to an alternating flux. Whereas a uniform flux is obtained by systematically taking

$$|j, l\rangle \xrightarrow{e^{i l k b}} |j+1, l\rangle \quad (\text{VI.80})$$

here one has (cf. figure VI.14)

$$|j, l\rangle \xrightarrow{e^{i l k b}} |j+1, l\rangle \quad \text{if } j \text{ is even} \quad (\text{VI.81})$$

since this corresponds to a transition $|g\rangle \rightarrow |e\rangle$ (similar to what we saw above) and

$$|j, l\rangle \xrightarrow{e^{-i l k b}} |j+1, l\rangle \quad \text{if } j \text{ is odd} \quad (\text{VI.82})$$

corresponding to a transition $|e\rangle \rightarrow |g\rangle$. The sum of the phase accumulated when going around a plaquette thus has a sign that alternates from one column to the next.

This alternating-flux lattice exhibits interesting properties when interactions are taken into account (Möller & Cooper 2010), but it does not correspond to the simulation of homogeneous magnetism being sought. It

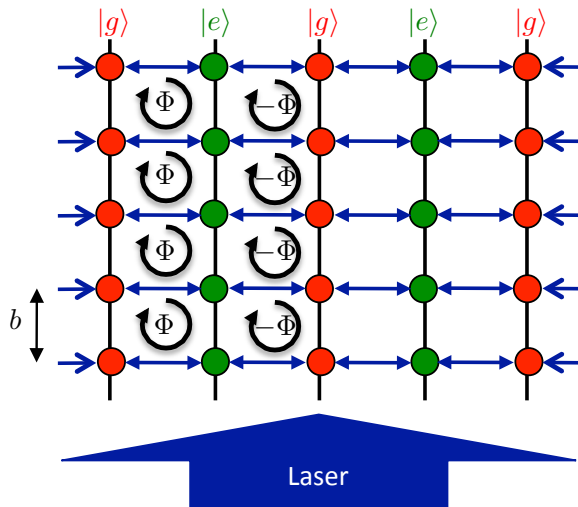


Figure VI.14. Realization of a square lattice with alternating flux: a single light beam is used to generate laser-assisted tunneling.

remains to “rectify” the magnetic field in order to obtain a flux of the same sign on each plaquette.

4-3 Flux rectification

Several techniques have been proposed for going from the alternating magnetic flux of figure VI.14 to a homogeneous flux. The initial proposal by Jaksch & Zoller (2003) made use of applying an additional linear potential to lift the degeneracy between the various possible $|e\rangle \rightarrow |g\rangle$ transitions. A slightly different version, better suited to alkaline-earth atoms, which are the most promising for this type of study, was proposed by Gerbier & Dalibard (2010). Its principle is sketched in figure VI.15. By means of an optical super-lattice along the x direction, with twice the period of the main lattice, the energy of the internal states $|g\rangle$ and $|e\rangle$ is modulated between two values, corresponding to the filled and open disks of figure VI.15. The various $|e\rangle \rightarrow |g\rangle$ transitions are provided by light beams

propagating along $+\mathbf{u}_y$ or $-\mathbf{u}_y$ and of suitably chosen frequency, to ensure that the phase accumulated around each plaquette remains everywhere the same.

5 Final remarks

Is it useful to bring several internal states into play?

We have explored in this section two possible approaches for realizing a Hofstadter-type Hamiltonian. The first does not use transitions between atomic internal states and consists in temporally modulating certain parameters of the lattice; the phase of this modulation will be transferred to the atoms and play the role of an Aharonov–Bohm phase. The second approach uses transitions between internal states induced by auxiliary beams. When an atom absorbs one of these auxiliary photons, the phase of the beam imprints itself on the atomic wave function, which generates the effect sought.

Let us now compare the advantages and drawbacks of these two methods. The first approach does not make use of light beams close to resonance, and thus in principle avoids any heating problem related to the phenomenon of spontaneous photon emission. On the other hand, it imposes fairly strong constraints on the amplitude of the achievable tunneling coefficients. To quantify these constraints, let us introduce the gap Δ between the lowest band with which one wishes to work and the first excited band. Our entire analysis has been based on a single-band calculation, and it is thus valid only if the transfer rate to the excited band is negligible. For this, the energy shift $\hbar\Omega_0$ between the two Wannier functions localized on two adjacent sites must be small compared to the gap between the two bands:

$$\hbar\Omega_0 \ll \Delta. \quad (\text{VI.83})$$

We saw that, to efficiently induce tunneling between adjacent sites, one had to choose the modulation frequency Ω close to Ω_0 . This frequency Ω characterizes the fast part of the evolution of the wave function, and the renormalized tunneling

$$\bar{J} = J \mathcal{J}_1(\xi) e^{i\phi} \quad (\text{VI.84})$$

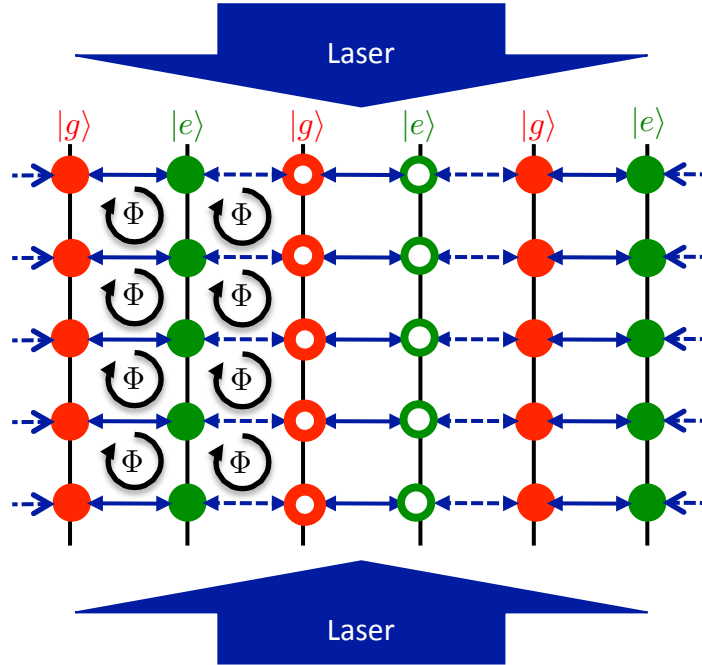


Figure VI.15. Simulation of a uniform flux with atoms in two internal states $|g\rangle$ and $|e\rangle$ trapped in an optical lattice. A super-lattice in the x direction modulates the energy of the $|g\rangle$ states between two values, represented by red disks alternately filled and open. The same holds for the $|e\rangle$ states. This generates 4 different transition frequencies for $|e\rangle \rightarrow |g\rangle$. Two of these transitions, marked by a solid double arrow, are induced by the light beam coming from below; the other two, marked by a dashed double arrow, are induced by the light beam coming from above. In this way, the sign of the phase accumulated on each plaquette is constant, which corresponds to the desired uniform flux.

characterizes, for its part, the slow part of this evolution. In the end, we thus arrive at the hierarchy of inequalities

$$|\bar{J}| \leq J \ll \Omega \approx \Omega_0 \ll \Delta. \quad (\text{VI.85})$$

The detuning Δ cannot be taken very large compared to the recoil energy, otherwise the tunneling effect J becomes negligible. The order of magnitude of the tunneling coefficients $\bar{J}$ that can be achieved by this method is thus, by construction, small.

For alkaline-earth atoms, the use of internal states can be done without risk of heating by spontaneous emission, since the lifetime of the excited state $|e\rangle$ is very long compared to any realistic duration of an experiment. The main constraint is therefore not to populate an excited band in the transition $|g\rangle \rightarrow |e\rangle$, which requires taking a Rabi frequency κ small compared to $\Delta/\hbar$. The tunneling coefficient obtained is then of order $\hbar\kappa$, with a reduction coefficient that involves the overlap between the Wannier functions for $|g\rangle$ and $|e\rangle$. We thus arrive at the series of inequalities:

$$\bar{J} \leq \hbar\kappa \ll \Delta. \quad (\text{VI.86})$$

This series is less restrictive than (VI.85), and one can thus expect potentially larger tunneling coefficients for this approach, which takes advantage of several internal states.

Let us finally note that we have not taken into account here many-particle effects; these can lead to notably different effects, depending on whether all the atoms are in the same internal state or whether several states are occupied, with the possibility of inelastic collisions between atoms.

Summary table

Over the past three lectures we have explored a number of methods for simulating orbital magnetism or spin-orbit coupling with neutral atoms. As we announced at the outset, these methods can be classified according to at least two criteria: (i) does one use several internal states or not? (ii) is the Hamiltonian time-dependent or not? We have summarized these different approaches in table VI.16. Each box of this table has been illustrated,

	Time-independent Hamiltonians	Time-dependent Hamiltonians frequency Ω
No use of internal states	Rotation with conserved angular momentum	$\Omega \sim \omega_c$: forced rotation with a rotating stirrer $\Omega \gg \omega_c$: shaken lattices
Use of internal states	Berry phase Flux lattices Laser-assisted tunneling Spin-orbit coupling	Spin-orbit coupling

Figure VI.16. Classification of procedures for generating artificial orbital magnetism (cyclotron angular frequency ω_c) or a spin-orbit coupling.

sometimes in several ways, in these lectures⁹. Depending on the atomic species considered and the phenomena sought, one may favor one or the other of these approaches, none of which has yet established itself as a universal strategy for studying the full range of phenomena related to orbital magnetism.

⁹with the exception of the box in the bottom right. For this one, see for example Xu, You, et al. (2013), Anderson, Spielman, et al. (2013), and Goldman & Dalibard (2014).

Chapter VII

Rotating condensates

In this final chapter we will be interested in the combined effect of an artificial magnetic field and interactions. This is an extremely vast subject that gives rise to numerous phenomena in condensed-matter physics, arising from both spin magnetism and orbital magnetism: ferro- and antiferromagnetism, the quantum Hall effect, superconductivity and the Meissner effect, spintronics, topological insulators and superconductors, etc. We will only address a very small part of this vast field here, focusing on the behavior of a superfluid in the presence of a uniform magnetic field.

The system we will consider is a condensate of bosons with repulsive interactions. We will start from the equivalence

Orbital magnetism $\longleftrightarrow$ Rotation

to study the behavior of a Bose gas placed in a rotating harmonic trap. This seemingly very simple system will allow us to introduce a series of important notions: mean-field approximation, critical frequency for the nucleation of a vortex, Abrikosov lattice, transition to strongly correlated states. Phenomena similar to those we will describe here are also expected for interacting Fermi gases, if these gases are cold enough to be described by a macroscopic wave function.

Here we will be interested in the case of a repulsive interaction between atoms, which represents the most interesting case in practice. We will restrict ourselves to a contact interaction, and we refer the reader to the review article by Cooper (2008) for a description of the phenomena likely to appear for a long-range interaction such as the dipole-dipole interaction.

We will begin this chapter with a brief presentation of the description of the interactions present in a cold-atom gas, before moving on to the description of a condensate in the mean-field regime. We will see how the notion of the lowest Landau level naturally appears in this context, and we will conclude with a description of the transition toward strongly correlated states, which has not yet been observed experimentally for these cold-atom gases. We will carry out this entire presentation for a two-dimensional gas, which makes it possible to somewhat simplify the mathematical treatment while retaining the essential phenomena. Numerous references to the phenomena discussed here can be found in the review articles by Bloch, Dalibard, et al. (2008), Cooper (2008), and Fetter (2009).

1 Interactions in a cold gas

1-1 Contact interaction

In what follows we will restrict ourselves to a gas of particles interacting via a short-range potential. More precisely, for two atoms separated by a distance r , the interaction potential consists of (cf. figure VII.1):

- an attractive part due to van der Waals forces

$$U_{\text{vdW}}(r) = -\frac{C_6}{r^6} \quad (\text{VII.1})$$

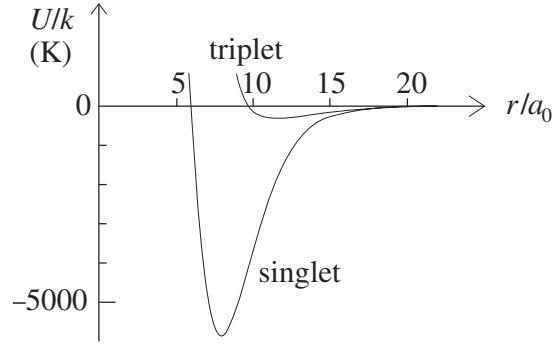


Figure VII.1. Simplified representation of the interaction potential between two rubidium atoms separated by the distance r . The unit of length is the Bohr radius $a_0 = 0.53 \text{ \AA}$ (figure taken from Pethick & Smith (2008)).

which dominates when r is large enough for the tunneling of an electron from one atom to the other to be negligible (typically $r > 8 \text{ \AA}$),

- an intermediate part, attractive or repulsive, corresponding to the covalent bond (hopping of the electron from one atom to the other), giving rise to the singlet and triplet potentials.
- a repulsive part at very short distance ($r < 4 \text{ \AA}$) due to the electrostatic repulsion of the electronic cores.

At very low energy, the scattering amplitude characterizing the collision between two atoms with relative wave vector k is isotropic and nearly independent of energy (s -wave collision). The collision is described by a single number, the *scattering length* a , from which the various relevant quantities are deduced. For example, for a gas of bosons all prepared in the same internal state, the collision cross section is $\sigma = 8\pi a^2$.

Since only the scattering length matters for describing interactions in a cold-atom gas, there is no need to know the exact shape of the potential shown in figure VII.1. Any other short-range potential leading to the same scattering length will give rise to the same physics. One therefore traditionally uses a very simple mathematical form, a contact interaction,

to model the real three-dimensional potential:

$$U^{(3D)}(\mathbf{r}) = \frac{4\pi\hbar^2 a}{M} \delta^{(3D)}(\mathbf{r}), \quad (\text{VII.2})$$

Remark: the pseudo-potential. Outside the one-dimensional case, a potential proportional to the Dirac distribution presents singularities that can lead to mathematical difficulties. A slightly more complicated version of the three-dimensional contact interaction makes it possible to avoid these difficulties; this is the pseudo-potential (Huang 1987), defined by its action on a function $\psi(\mathbf{r})$:

$$U^{(3D)}(\mathbf{r}) \psi(\mathbf{r}) = \frac{4\pi\hbar^2 a}{M} \delta(\mathbf{r}) \frac{\partial}{\partial r} (r\psi(\mathbf{r})). \quad (\text{VII.3})$$

In practice, we will not need to resort to this pseudo-potential for the mean-field treatment that follows.

What is a cold gas for collisions? We mentioned that the cross section becomes isotropic for a sufficiently low collision energy. Let us give a precise criterion for this approximation. When one performs a partial-wave expansion to treat the collision between particles, one finds, for each partial wave of angular momentum ℓ , the centrifugal barrier

$$U_{\text{centrif.}}^{(\ell)}(r) = \frac{\hbar^2 \ell(\ell+1)}{2m_r r^2}, \quad (\text{VII.4})$$

where $m_r = M/2$ is the reduced mass of the pair of atoms. For s -wave collisions, this barrier is zero. On the other hand, as soon as one considers $\ell \neq 0$, this barrier prevents the atoms from approaching one another, and it thus has an effect opposite to the attractive van der Waals potential.

Let us look at the sum of these potentials for the partial wave¹ $\ell = 2$ (cf. figure VII.2):

$$U_{\text{tot}}^{(\ell=2)}(r) = U_{\text{vdW}}(r) + U_{\text{centrif.}}^{(\ell=2)}(r) = -\frac{C_6}{r^6} + \frac{3\hbar^2}{m_r r^2}. \quad (\text{VII.5})$$

¹For polarized bosons, the partial wave $\ell = 1$ is forbidden, since it corresponds to a wave function antisymmetric under exchange of the two particles.

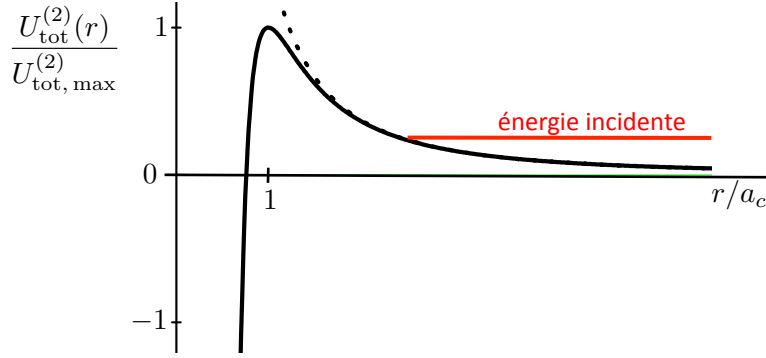


Figure VII.2. Solid line: sum of the van der Waals potential and the centrifugal potential for $\ell = 2$; dotted line: centrifugal potential alone. For an incident energy much lower than the height $U_{\text{tot,max}}^{(2)}$, the particles hardly feel the van der Waals potential.

This potential is maximal at $r = a_c/2^{1/4}$, where a_c is the characteristic length associated with the van der Waals interaction:

$$a_c = \left(\frac{2m_r C_6}{\hbar^2} \right)^{1/4}. \quad (\text{VII.6})$$

The height of the barrier at this point is

$$U_{\text{tot,max}}^{(2)} = \sqrt{8} \frac{\hbar^2}{m_r a_c^2}. \quad (\text{VII.7})$$

If the incident energy of the particles is well below this height, i.e.,

$$k_B T \ll \frac{\hbar^3}{(m_r^3 C_6)^{1/2}}, \quad (\text{VII.8})$$

their minimal distance of approach for this partial wave will remain much larger than a_c . This means that the particles will essentially be sensitive to the centrifugal potential, and not to the van der Waals potential.

For this partial wave $\ell = 2$ (and for all the following ones, which have an even larger centrifugal barrier), everything thus happens as if the van

der Waals interaction did not exist. Only the s -wave, corresponding to $\ell = 0$, which contains no centrifugal potential, contributes. Now, s -wave scattering is by definition isotropic, hence the result. For alkali atoms, one finds that the right-hand side of inequality (VII.8) is of order a hundred microkelvins, or even more, and the approximation of s -wave collisions is excellent in the regime of quantum degeneracy experimentally accessible (around or below a microkelvin).

1-2 Mean-field approximation

Let us now consider an assembly of N atoms placed in an external potential $V(\mathbf{r})$ and interacting via the potential $\sum U(\mathbf{r}_i - \mathbf{r}_j)$, where $\mathbf{r}_i$ and $\mathbf{r}_j$ are the coordinates of two atoms. The total Hamiltonian of the system is thus written:

$$\hat{H}^{(N)} = \sum_{j=1}^N \left(\frac{\hat{p}_j^2}{2M} + V(\mathbf{r}_j) \right) + \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} U(\mathbf{r}_i - \mathbf{r}_j) \quad (\text{VII.9})$$

We will seek the ground state of this system in an approximate way, corresponding to the physical idea that all the particles are condensed and described by the same wave function. This amounts to using a mean-field approximation; we neglect any correlation between atoms and write the ground state in the form

$$\Psi(\mathbf{r}_1, \dots, \mathbf{r}_N) = \psi(\mathbf{r}_1) \dots \psi(\mathbf{r}_N), \quad (\text{VII.10})$$

where the macroscopic wave function ψ is normalized:

$$\int |\psi(\mathbf{r})|^2 d^3r = 1. \quad (\text{VII.11})$$

Once the function $\psi(\mathbf{r})$ is chosen, one can evaluate the average energy of the system

$$E^{(N)}[\psi] = \langle \Psi | \hat{H}^{(N)} | \Psi \rangle \quad (\text{VII.12})$$

The N kinetic-energy and potential-energy terms all give the same contribution

$$\int \left(\frac{\hbar^2}{2M} |\nabla \psi(\mathbf{r})|^2 + V(\mathbf{r}) |\psi(\mathbf{r})|^2 \right) d^3r, \quad (\text{VII.13})$$

and the $N(N-1)/2$ interaction terms likewise all have the same expression

$$\iint U(\mathbf{r}_1 - \mathbf{r}_2) |\psi(\mathbf{r}_1)|^2 |\psi(\mathbf{r}_2)|^2 d^3r_1 d^3r_2 = \frac{4\pi\hbar^2 a}{M} \int |\psi(\mathbf{r})|^4 d^3r. \quad (\text{VII.14})$$

Taking $N-1 \approx N$, we thus obtain, in this mean-field approximation, the energy per particle $E[\psi] = E^{(N)}[\psi]/N$:

$$E[\psi] = \int \left(\frac{\hbar^2}{2M} |\nabla\psi(\mathbf{r})|^2 + V(\mathbf{r})|\psi(\mathbf{r})|^2 + \frac{2\pi\hbar^2 a N}{M} |\psi(\mathbf{r})|^4 \right) d^3r \quad (\text{VII.15})$$

which must be minimized by choosing the appropriate value of ψ , while taking the normalization constraint into account.

Gross–Pitaevskii equation. Minimizing the energy (VII.15) under the constraint that the wave function must be normalized (VII.11) is a variational problem that can be solved by the method of Lagrange multipliers. The states that make (VII.15) extremal are the solutions of the Gross–Pitaevskii equation

$$-\frac{\hbar^2}{2M} \Delta\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) + \frac{4\pi\hbar^2 a N}{M} |\psi(\mathbf{r})|^2 \psi(\mathbf{r}) = \mu \psi(\mathbf{r}), \quad (\text{VII.16})$$

where the *chemical potential* μ is the Lagrange multiplier introduced to take into account the constraint on the norm. It corresponds physically to the increase $E^{(N+1)} - E^{(N)}$ of the ground-state energy when one particle is added to the N -body system.

Healing length ξ . An important point of this mean-field description is the emergence of a characteristic length associated with the interactions, the *healing length* ξ . To show how this length arises, let us consider a condensate confined in the half-space $x > 0$, with a wall in the plane $x = 0$ that forces the wave function to vanish on this plane. The repulsive interactions between atoms will favor states in which the gas occupies as much space as possible; for these states, the wave function takes significant values at distances arbitrarily close to the wall, and thus increases slowly from the node at $x = 0$. On the other hand, the kinetic-energy term favors wave functions that vary slowly in space. The healing length emerges as the best compromise between these two constraints.

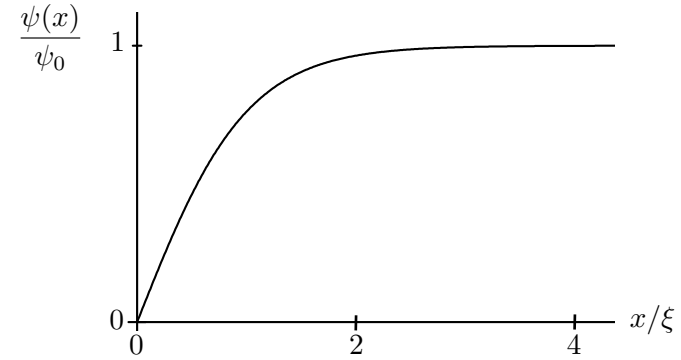


Figure VII.3. Variation (VII.17) of the wave function near a wall, modeled as the flat surface $x = 0$ on which $\psi(x)$ must vanish. The characteristic length ξ , called the healing length is related to the chemical potential μ by $\xi = \hbar/\sqrt{M\mu}$ (a relation valid both in 3D and in 2D).

One can find the exact solution of the Gross–Pitaevskii equation (VII.16) in the case of interest to us

$$\psi(x) = \psi_0 \tanh(x/\xi), \quad (\text{VII.17})$$

with

$$\xi = \frac{\hbar}{\sqrt{M\mu}} = \frac{1}{\sqrt{4\pi a \rho^{(3D)}}}, \quad \mu = \frac{4\pi\hbar^2 a}{M} \rho^{(3D)}, \quad \rho^{(3D)} = N\psi_0^2. \quad (\text{VII.18})$$

This wave function indeed vanishes at $x = 0$ and tends to the constant ψ_0 for $x \gg \xi$.

1-3 Passage to two dimensions

We will now focus on the properties of the atomic gas in the xy plane. This choice is motivated by the fact that the magnetism induced by a field $\mathbf{B}$ parallel to the z axis manifests itself essentially in this plane. This restriction will allow us to notably simplify the calculations and notation, while retaining the physically important elements.

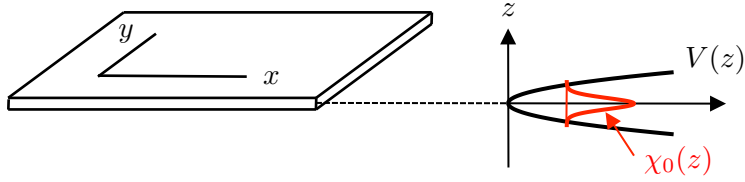


Figure VII.4. Confinement along the z direction by a harmonic potential. The state of the gas along z is assumed to be “frozen” and described by the wave function $\chi_0(z)$, of extent a_z . The interactions in the xy plane are then described by the dimensionless number $g \propto a/a_z$ (cf. VII.22), where a is the 3D scattering length.

Mathematically, this restriction to the xy plane is carried out by seeking the wave functions ψ that minimize the energy (VII.15) in the factorized form

$$\psi(x, y, z) = \phi(x, y) \chi_0(z), \quad (\text{VII.19})$$

and assuming that $\chi_0(z)$ is frozen. Physically, such a regime can be achieved by applying, along z , a very strong confinement so that the atoms only occupy the ground state of the Hamiltonian describing the motion along this direction (figure VII.4).

We will further assume that the confining potential in the xy plane is harmonic and isotropic, with angular frequency ω . We thus reduce to the mean-field energy:

$$E[\phi] = \int \left(\frac{\hbar^2}{2M} |\nabla \phi(\mathbf{r})|^2 + \frac{1}{2} M \omega^2 r^2 |\phi(\mathbf{r})|^2 + \frac{\hbar^2}{2M} N g |\phi(\mathbf{r})|^4 \right) d^2r, \quad (\text{VII.20})$$

where the (dimensionless) interaction constant g is defined by

$$g = 4\pi a \int |\chi_0(z)|^4 dz. \quad (\text{VII.21})$$

If the confinement along z is harmonic with angular frequency ω_z , and if $\chi_0(z) \propto \exp(-z^2/2a_z^2)$ represents the ground state of this oscillator, then

$$g = \sqrt{8\pi} \frac{a}{a_z}, \quad (\text{VII.22})$$

where $a_z = \sqrt{\hbar/M\omega_z}$. Consider, for example, a rubidium gas confined by a

trap of frequency $\omega_z/2\pi = 5$ kHz; one finds $a_z = 150$ nm, which, combined with $a = 5.1$ nm, leads to a coupling $g = 0.17$.

Effective two-dimensional interaction. The two-dimensional mean-field energy given in (VII.20) can be formally obtained by taking, for the interaction potential between two particles,

$$U^{(2D)}(\mathbf{r}) = \frac{\hbar^2}{M} g \delta^{(2D)}(\mathbf{r}). \quad (\text{VII.23})$$

In the general case, this two-dimensional Dirac distribution can lead to mathematical singularities; to describe a two-dimensional collision process in quantum physics, one must in principle introduce an energy-dependent coupling coefficient [see for example Adhikari (1986), Petrov & Shlyapnikov (2001), and Olshanii & Pricoupenko (2002)]. However, these singularities do not come into play in the mean-field treatment we consider here, nor in the beyond-mean-field treatment restricted to the lowest Landau level, which we will address in paragraph 4 (Haldane 1983); in both cases, one can represent the interaction between particles by the two-dimensional contact potential (VII.23), characterized by the constant coupling coefficient g , without introducing any mathematical pathology.

1-4 Thomas-Fermi regime in 2D

To make progress in the study of the equilibrium state of the gas, let us begin by noting that the stability of the cloud results from two competing phenomena present in the expression for the energy (VII.20):

- The trapping term $M\omega^2 r^2/2$ favors a function ϕ localized near the origin, so as to minimize the potential energy.
- The two kinetic-energy and interaction-energy terms have the opposite effect: these two energies decrease as the extent of the function ϕ increases.

The trapping term thus always plays an essential role in the equilibrium. Its effect will be predominantly counterbalanced either by the kinetic energy or by the interaction energy. To determine which of these two energies

is dominant in ensuring equilibrium, let us introduce the size R characterizing the extent of ϕ . Since ϕ is normalized, we have $\phi \sim 1/R$ over the region where it takes significant values. We deduce

$$\frac{\hbar^2}{M} \int |\nabla \phi(\mathbf{r})|^2 d^2r \sim \frac{\hbar^2}{MR^2}, \quad \frac{\hbar^2}{M} Ng \int |\phi(\mathbf{r})|^4 d^2r \sim \frac{\hbar^2}{MR^2} Ng. \quad (\text{VII.24})$$

Consequently, if $Ng \ll 1$, it is the kinetic energy that counterbalances the potential energy, and one can ignore the interactions. In the opposite case $Ng \gg 1$, it is the repulsive interaction between atoms that opposes the trapping force, and the kinetic energy is negligible. This second case, which corresponds to the *Thomas–Fermi regime*, is easily reached in practice; taking the value $g \sim 0.2$ given above, it suffices to place about ten atoms in the trap.

If the kinetic energy is neglected, (VII.20) simplifies to give

$$Ng \gg 1: \quad E[\phi] \approx \frac{1}{2} M \omega^2 \int r^2 |\phi|^2 + \frac{\hbar^2}{2M} Ng \int |\phi|^4, \quad (\text{VII.25})$$

and the associated Gross-Pitaevskii equation becomes an algebraic equation

$$\frac{1}{2} M \omega^2 r^2 \phi(\mathbf{r}) + \frac{\hbar^2}{M} Ng |\phi(\mathbf{r})|^2 \phi(\mathbf{r}) = \mu \phi(\mathbf{r}). \quad (\text{VII.26})$$

whose solution is simple. Introducing the Thomas-Fermi radius R_{TF}

$$\mu = \frac{1}{2} m \omega^2 R_{\text{TF}}^2; \quad (\text{VII.27})$$

one obtains

$$\begin{aligned} Ng |\phi(\mathbf{r})|^2 &= \frac{1}{2a_{\perp}^4} (R_{\text{TF}}^2 - r^2) & \text{if } r < R \\ \phi(\mathbf{r}) &= 0 & \text{if } r \geq R. \end{aligned} \quad (\text{VII.28})$$

where we have set $a_{\perp} = (\hbar/M\omega)^{1/2}$. To complete our analysis, it remains to express the fact that the function $\phi(\mathbf{r})$ is normalized, which provides a relation between the Thomas–Fermi radius and the parameter Ng :

$$R_{\text{TF}} = a_{\perp} \left(\frac{4Ng}{\pi} \right)^{1/4}, \quad \mu = \hbar \omega \left(\frac{Ng}{\pi} \right)^{1/2}. \quad (\text{VII.29})$$

A quantity useful for what follows is the surface density $\rho(\mathbf{r})$, and in particular its value at the center of the condensate:

$$\rho(0) = N |\phi(0)|^2 = \sqrt{\frac{N}{\pi g}} a_{\perp}^{-2}. \quad (\text{VII.30})$$

One thus observes that two effects occur when the number of atoms in the condensate is increased:

- the radius increases as $N^{1/4}$, and hence the area as $\sqrt{N}$.
- the density at the center also increases as $\sqrt{N}$.

Remark. The relations between the chemical potential, the healing length, and the Thomas–Fermi radius:

$$\mu = \frac{\hbar^2}{M\xi^2}, \quad \mu = \frac{1}{2} M \omega^2 R_{\text{TF}}^2 = \hbar \omega \frac{R_{\text{TF}}^2}{2a_{\perp}^2}, \quad (\text{VII.31})$$

can be combined to give

$$\xi = a_{\perp} \left(\frac{\pi}{Ng} \right)^{1/4}, \quad \xi R_{\text{TF}} = \sqrt{2} a_{\perp}^2. \quad (\text{VII.32})$$

For a trap of given frequency, and hence given $a_{\perp}$, the healing length and the Thomas-Fermi radius vary in opposite directions as the number of atoms is increased: ξ decreases and R_{TF} increases, their product remaining constant.

2 Vortices in a condensate

We now consider a condensate confined in an isotropic harmonic trap of angular frequency ω in the xy plane, and set into rotation with a stirrer rotating at angular frequency Ω . The single-particle Hamiltonian is written, in the rotating frame,

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2M} + \frac{1}{2} M \omega^2 \hat{\mathbf{r}}^2 - \Omega \hat{L}_z \quad (\text{VII.33})$$

In what follows we will be interested in the ground state of this condensate in the presence of interactions, under the assumption that the latter can be described by a mean-field interaction. It will thus be a matter of finding the function $\phi(x, y)$ that minimizes the energy functional associated with (VII.33):

$$E[\phi] = \int \left(\frac{\hbar^2}{2M} |\nabla \phi|^2 + \frac{1}{2} M \omega^2 r^2 |\phi|^2 - \Omega \phi^* (\hat{L}_z \phi) + \frac{\hbar^2}{2M} N g |\phi|^4 \right) d^2 r. \quad (\text{VII.34})$$

2-1 The appearance of the first vortex

The notion of a vortex will play a crucial role in the search for the ground state in the presence of a gauge field. We already encountered this notion when we studied, in chapter 2, the eigenstates of the single-particle Hamiltonian in an isotropic trap. We showed that the ground state and the first two excited states were given by

$$\text{Ground state:} \quad \psi_0(\mathbf{r}) = e^{-r^2/2a_\perp^2}, \quad (\text{VII.35})$$

$$\text{First excited level:} \quad \psi_\pm(\mathbf{r}) = r e^{\pm i\varphi} e^{-r^2/2a_\perp^2} \quad (\text{VII.36})$$

The two states that form a basis for the first excited level carry a nonzero angular momentum, namely $\pm\hbar$. They possess all the characteristics of the vortices we will encounter in what follows:

- The center of a vortex is always a point at which the density $\rho(\mathbf{r}) = |\phi(\mathbf{r})|^2$ vanishes.
- Along a closed contour surrounding the center of the vortex, the phase of $\phi(\mathbf{r})$ evolves continuously, giving a winding of $\pm 2\pi$, or a multiple of this quantity in the case of a multiply charged vortex.
- For a single vortex at the center of a condensate, the velocity field $\mathbf{v}(\mathbf{r})$ deduced from the phase $\theta(\mathbf{r})$ of the wave function

$$\phi(\mathbf{r}) = |\phi(\mathbf{r})| e^{i\theta(\mathbf{r})} \quad \rightarrow \quad \mathbf{v}(\mathbf{r}) = \frac{\hbar}{M} \nabla \theta(\mathbf{r}), \quad (\text{VII.37})$$

varies as $1/r$. This property also exists for a classical fluid, but the distinctive feature of a condensate is the quantization of the velocity

circulation: on a contour surrounding the vortex, one has

$$\mathbf{v}(\mathbf{r}) = \pm \frac{\hbar}{Mr} \mathbf{u}_\varphi, \quad \oint \mathbf{v}(\mathbf{r}) \cdot d\mathbf{r} = \pm \frac{\hbar}{M} \quad (\text{VII.38})$$

for a vortex of charge 1.

The important modification that will appear in the presence of interactions is the size of the vortex core. Whereas for the states (VII.36) the core size (typically the full width at half maximum of the central density hole) is of order $a_\perp$, the equilibrium core size of a vortex in the presence of interactions is smaller, of order ξ (this will no longer be true once the ground state is restricted to the LLL, see § 3). The appearance of the healing length in this context is not surprising. Imposing the presence of a vortex at a given point amounts to requiring the wave function to vanish at that point. Moving away from the core, the surface density $\rho(\mathbf{r})$ increases to recover, far away, its value in the absence of a vortex. The equilibrium core size, which results from a compromise between kinetic energy and interaction energy, is thus naturally given [up to a numerical factor, see for example Castin & Dum (1999)] by the healing length.

It is interesting to evaluate the *number of missing atoms* δN in a vortex core in this two-dimensional case. Using (VII.30-VII.32), one finds

$$\delta N \sim \rho(0) \pi \xi^2 = \frac{\pi}{g}. \quad (\text{VII.39})$$

For the physical situation considered in § 1-3 for rubidium atoms ($g = 0.17$), one finds $\delta N \sim 20$. The atom deficit associated with the presence of a vortex is thus small compared to the typical number of atoms in a condensate (10^3 to 10^6). The situation becomes even more striking when the strength of the interactions is increased to reach a value of the coupling g exceeding unity. The atom deficit δN then becomes smaller than 1, which means that a vortex cannot be detected by the hole it creates in the gas *in situ*. This is what occurs in liquid-helium films, for example.

Since a state with a single centered vortex corresponds to an angular momentum of $\hbar$ per particle, it lowers the energy of at least one of the terms of $E[\phi]$ given in (VII.34), namely $-\Omega \langle L_z \rangle$. One can thus expect to see one or more vortices appear in the ground state of the rotating gas.

However, there is an additional kinetic energy² associated with the vortex velocity field $v(\mathbf{r})$:

$$\Delta E^{(N)} = \frac{M}{2} \int \rho(\mathbf{r}) v^2(\mathbf{r}) d^2r, \quad (\text{VII.40})$$

which we must evaluate to determine when the single-vortex state becomes energetically favorable compared to the vortex-free state. The density profile $\rho(\mathbf{r})$ is essentially unchanged from its value in the absence of a vortex, except in the immediate vicinity of the core, i.e., for $r \lesssim \xi$. For the azimuthal velocity field $v(\mathbf{r})$ (VII.38), the increase in kinetic energy per particle is

$$\Delta E = \frac{\Delta E^{(N)}}{N} \approx \frac{M}{2N} \rho(0) \int_{\xi}^{R_{\text{TF}}} \left(1 - \frac{r^2}{R_{\text{TF}}^2}\right) \frac{\hbar^2}{M^2 r^2} 2\pi r dr. \quad (\text{VII.41})$$

The integral can be computed simply to give

$$\Delta E \approx \hbar\omega \sqrt{\frac{\pi}{Ng}} \ln \frac{R_{\text{TF}}}{e\xi}. \quad (\text{VII.42})$$

The Thomas–Fermi approximation is valid in the case $Ng \gg 1$; this increase in energy per particle is thus small³ compared to $\hbar\omega$, whereas the energy per particle ($\sim \mu$) is, for its part, large compared to $\hbar\omega$ [see for example (VII.31)].

The kinetic-energy cost of a vortex is thus relatively small, and, thanks to the term in $-\Omega L_z$, a weak rotation suffices to make its creation energetically favorable. More precisely, since a centered vortex corresponds to an angular momentum $\hbar$, the energy change $\Delta E - \hbar\Omega$ becomes negative when Ω exceeds the critical value Ω_c given by (Baym & Pethick 1996; Dalfovo & Stringari 1996):

$$\frac{\Omega_c}{\omega} \approx \sqrt{\frac{\pi}{Ng}} \ln \frac{R_{\text{TF}}}{e\xi} \ll 1. \quad (\text{VII.43})$$

²One can also estimate the increase in potential energy of the cloud due to the transfer of particles initially at a distance $r \lesssim \xi$ from the center toward the outer layers, and check that it is negligible compared to the increase in kinetic energy computed here.

³The argument of the logarithm is at most of order a thousand, and the logarithm itself is thus of order a few units.

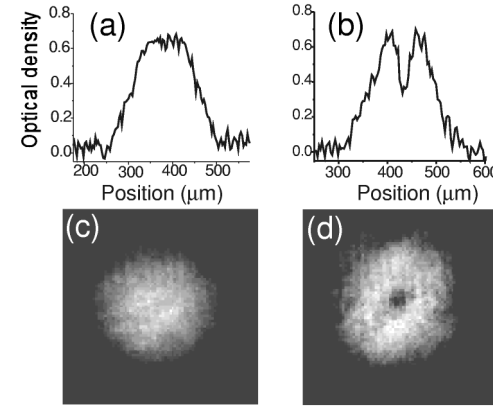


Figure VII.5. Vortex observed at the ENS in a rotating rubidium condensate (3D) (Madison, Chevy, et al. 2000a). These images were taken after a time of flight, which increases the size of the vortex core, making it observable (for a theoretical description of the effect of this time of flight, see for example Dalfovo & Modugno (2000)). Experiments carried out with $\sim 10^5$ rubidium atoms in a trap such that $\omega/2\pi = 220$ Hz. Left, $\Omega/2\pi = 145$ Hz; right, $\Omega/2\pi = 152$ Hz.

First experiments. We have reported in figure VII.5 the first observations of vortices in rotating atomic condensates (Madison, Chevy, et al. 2000a). It is indeed found that, above a critical rotation frequency, a first vortex appears in the condensate. In this first experiment, the critical frequency Ω_c did not correspond to that predicted in (VII.43), but was systematically higher, roughly equal to $\omega/\sqrt{2}$. The quantitative explanation for this phenomenon lies in the fact that there is an energy barrier to overcome for a vortex to penetrate into a condensate. It is not enough for the final state with a vortex to have a lower energy than the vortex-free state; there must also exist a path for the vortex, initially outside the cloud, to penetrate into it. The existence of such a path is favored for $\Omega \approx \omega/\sqrt{2}$, due to a dynamical instability of the surface of the condensate around this rotation frequency (Sinha & Castin 2001), [see also Recati, Zambelli, et al. (2001), Lobo, Sinatra, et al. (2004) and the associated films on Yvan Castin’s personal webpage].

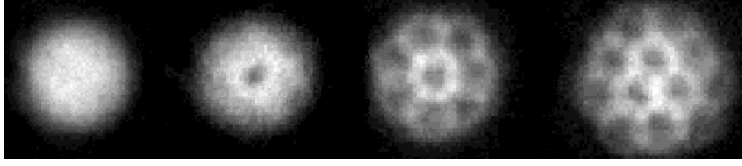


Figure VII.6. Increase in the number of vortices with the rotation frequency. Figure obtained at the ENS, taken from Madison, Chevy, et al. (2000c).

2-2 Feynman's argument

When the artificial magnetic field or the rotation frequency Ω increases, the number of vortices within the condensate at equilibrium grows, as can be seen in figure VII.6. Before studying this specifically quantum problem, it is useful to begin with the study of the equivalent classical problem. Let us thus consider a classical fluid (assumed incompressible for simplicity), of density ρ , confined in a circular box of radius R . The box rotates about its axis at angular frequency Ω , and we assume the walls are rough enough to set the fluid in motion. The search for the equilibrium state must thus be carried out in the rotating frame, since this frame is the only one in which the forces acting on the fluid are time-independent.

Let us work in two dimensions and look for the velocity field $\mathbf{v}(\mathbf{r})$ of the fluid in the laboratory, once equilibrium is reached. The kinetic energy of the fluid in the laboratory is

$$E_c^{(\text{lab})} = \frac{1}{2} M \rho \int \mathbf{v}^2(\mathbf{r}) \, d^2r \quad (\text{VII.44})$$

and the energy in the rotating frame is:

$$E_c^{(\text{rt})} = E_c^{(\text{lab})} - \Omega L_z, \quad \text{with} \quad L_z = \rho \int \mathbf{r} \times \mathbf{v}(\mathbf{r}) \, d^2r. \quad (\text{VII.45})$$

This energy can be cast in the form

$$E_c^{(\text{rt})} = \frac{1}{2} M \rho \int (\mathbf{v}(\mathbf{r}) - \boldsymbol{\Omega} \times \mathbf{r})^2 \, d^2r - \frac{1}{2} J \Omega^2, \quad (\text{VII.46})$$

with $\boldsymbol{\Omega} = \Omega \mathbf{u}_z$. The quantity $J = M \int r^2 d^2r$ denotes the moment of inertia of the disk. It is then immediate to see that the velocity field minimizing

(VII.46) is the rigid-rotation field:

$$\mathbf{v}(\mathbf{r}) = \boldsymbol{\Omega} \times \mathbf{r}. \quad (\text{VII.47})$$

This velocity field corresponds to a uniform vorticity (curl of the velocity field)

$$\nabla \times \mathbf{v}(\mathbf{r}) = 2 \boldsymbol{\Omega}. \quad (\text{VII.48})$$

Let us now return to the case of a superfluid described by a macroscopic wave function $\phi(\mathbf{r}) = |\phi(\mathbf{r})| e^{i\theta(\mathbf{r})}$. We already indicated that the velocity field at a point of nonzero density is

$$\mathbf{v}(\mathbf{r}) = \frac{\hbar}{M} \nabla \theta(\mathbf{r}). \quad (\text{VII.49})$$

We deduce that, away from a density zero, the vorticity vanishes, since the curl of a gradient is always zero.

Let us now consider a vortex centered at $\mathbf{r}_0$; the velocity field is azimuthal near this point:

$$\mathbf{v}(\mathbf{r}) \approx \frac{\hbar}{M |\mathbf{r} - \mathbf{r}_0|} \mathbf{u} + \dots, \quad (\text{VII.50})$$

which corresponds to the vorticity:

$$\nabla \times \mathbf{v}(\mathbf{r}) = \frac{2\pi\hbar}{M} \delta(\mathbf{r} - \mathbf{r}_0) + \dots \quad (\text{VII.51})$$

In a superfluid, the vorticity is thus concentrated at discrete points corresponding to the centers of the vortices, whereas it is diffuse in a classical fluid [cf. eq. (VII.48)].

To find the distribution of vortices in a superfluid at equilibrium, Feynman (1955) suggested using the correspondence principle and positing that the average vortex density ρ_v must lead, after smoothing, to a vorticity equal to that of a classical fluid for the same rotation. Comparing (VII.48) and (VII.51), this argument leads to

$$2\Omega = \frac{2\pi\hbar}{M} \rho_v \quad \Rightarrow \quad \rho_v = \frac{M\Omega}{\pi\hbar}. \quad (\text{VII.52})$$

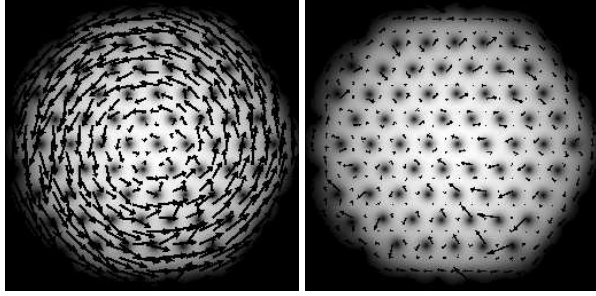


Figure VII.7. Figure taken from the article by Feder & Clark (2001), showing the density distribution and the velocity field of a condensate in the laboratory frame (left) and in the rotating frame (right). These results were obtained for a condensate of $2 \cdot 10^5$ rubidium atoms in a 3D trap with frequencies $\omega/2\pi = 8$ Hz and $\omega_z/2\pi = 5$ Hz. The rotation frequency is $\Omega = 0.95 \omega$.

This hypothesis is confirmed by numerical simulations; we show in figure VII.7 the result of a calculation by Feder, Clark, et al. (1999), showing the minimum-energy state of a rotating condensate. This state does indeed contain vortices arranged in a regular triangular lattice, similar to the Abrikosov lattice of vortices in type-II superconductors placed in a magnetic field. The average vortex density is in good agreement with Feynman's prediction (VII.52), and the velocity field correctly reproduces the rigid-rotation field (VII.47) (with, of course, deviations near the vortex cores).

Remark. We have restricted ourselves here to the case of charge-1 vortices, which is justified by a dynamical or thermodynamic stability analysis of higher-charge vortices: the latter are unstable, and it is always energetically preferable to split them into two (or more) unit-charge vortices (Castin & Dum 1999).

2-3 The equilibrium shape of the condensate

The equilibrium distribution of atoms in a rotating trap is obtained (within the mean-field approximation) by minimizing the total energy (VII.20), which we rewrite here in the rotating frame:

$$E[\phi] = E_c[\phi] + E_p[\phi] + E_{\text{int}}[\phi] - \Omega \langle L_z \rangle_\phi \quad (\text{VII.53})$$

with

$$E_c[\phi] = \frac{\hbar^2}{2M} \int |\nabla \phi|^2, \quad E_p[\phi] = \frac{1}{2} M \omega^2 \int r^2 |\phi|^2 \quad (\text{VII.54})$$

$$E_{\text{int}}[\phi] = \frac{\hbar^2}{2M} N g \int |\phi|^4, \quad \langle L_z \rangle_\phi = -i\hbar \int \phi^* (\mathbf{r} \times \nabla) \phi \quad (\text{VII.55})$$

It is now a matter of finding an approximate form for the function ϕ that minimizes the total energy (VII.53). For this, we will make the following assumption:

The size ξ of the vortices is small compared to the average distance between vortices ρ_v^{-2} .

This assumption allows us to neglect the density variation near the core in the estimate of the kinetic energy and the interaction energy, and to treat the function ϕ as a regular function. Thanks to Feynman's argument, we also know that the velocity field $\mathbf{v}(\mathbf{r})$ is given, to a good approximation, by the rigid-rotation velocity field (VII.47). We can then make the approximation:

$$\mathbf{v} \approx \boldsymbol{\Omega} \times \mathbf{r} \quad \Rightarrow \quad E_c[\phi] \approx \frac{1}{2} M \Omega^2 \int r^2 |\phi|^2, \quad (\text{VII.56})$$

$$\text{and} \quad -\Omega \langle L_z \rangle_\phi \approx -M \Omega^2 \int r^2 |\phi|^2. \quad (\text{VII.57})$$

Starting from these two assumptions, the energy (VII.53) can be rewritten

$$E[\phi] \approx \frac{1}{2} M (\omega^2 - \Omega^2) \int r^2 |\phi|^2 + \frac{\hbar^2}{2M} N g \int |\phi|^4 \quad (\text{VII.58})$$

We thus recover a standard energy functional within the Thomas–Fermi approximation, the only modification relative to (VII.25) being the substitution:

$$\omega^2 \longrightarrow \omega^2 - \Omega^2, \quad (\text{VII.59})$$

which corresponds to the reduction of the trap stiffness due to the centrifugal potential.

All the results obtained in the first paragraph concerning the equilibrium shape of a condensate in the Thomas-Fermi limit thus remain qualitatively valid: in particular, the average density profile of the condensate keeps its parabolic shape, but the radius R_{TF} increases

$$R_{\text{TF}} = a_{\perp} \left(\frac{4}{\pi} Ng \right)^{1/4} \rightarrow R_{\text{TF}} = a_{\perp} \left(\frac{4}{\pi} \frac{Ng}{1 - \Omega^2/\omega^2} \right)^{1/4}. \quad (\text{VII.60})$$

This radius thus grows indefinitely as Ω increases, and we recover the centrifugal limit: the equilibrium radius becomes infinite when Ω reaches the trapping value ω , with:

$$\Omega \rightarrow \omega : \quad R_{\text{TF}} \approx a_{\perp} \left(\frac{2}{\pi} \frac{Ng}{1 - \Omega/\omega} \right)^{1/4}. \quad (\text{VII.61})$$

2-4 Experimental observations

We have reported in figure VII.8a a spectacular observation made at MIT of a lattice of more than a hundred vortices [figure taken from the article by Abo-Shaeer, Raman, et al. (2001)]. This experiment provided a first test of the law governing the variation of the total number of vortices in the cloud

$$N_v = \rho_v \pi R_{\text{TF}}^2 \quad \text{with} \quad \rho_v = \frac{M\Omega}{\pi\hbar}, \quad (\text{VII.62})$$

as a function of the rotation frequency Ω . This prediction (adapted to the 3D case) is plotted as a dotted line in figure VII.8b, taken from Raman, Abo-Shaeer, et al. (2001). One finds in this figure that the number of vortices detected is systematically somewhat lower than expected, which the authors of Raman, Abo-Shaeer, et al. (2001) attribute to an imperfect equilibrium state of the condensate with the rotating stirrer.

Similar experiments have also been carried out with Fermi gases prepared in a superfluid state in a strongly interacting regime (BEC-BCS crossover). Vortex lattices have also been observed on either side of the unitary limit [see figure VII.9, taken from the article of Zwierlein, Abo-Shaeer, et al. (2005)].

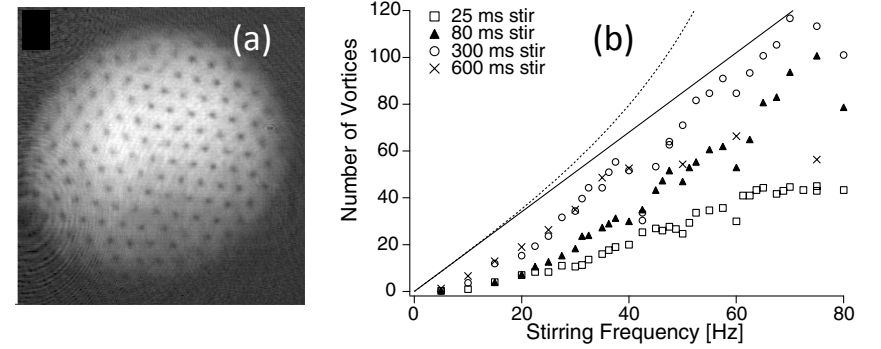


Figure VII.8. Figures obtained at MIT, taken from the articles by Abo-Shaeer, Raman, et al. (2001) and Raman, Abo-Shaeer, et al. (2001), showing a large vortex lattice for a 3D condensate of 50 million sodium atoms, in a trap of frequency $\omega/2\pi = 86$ Hz in the xy plane. The rotation frequency for panel a is 60 Hz. The solid line in panel b shows the number of vortices expected according to the law $\rho_v = M\Omega/\pi\hbar$, taking into account the increase in radius due to the centrifugal deformation.

3 Rotation and the lowest Landau level

When the rotation speed Ω is increased to approach the centrifugal limit $\Omega = \omega$ and reach ever-larger artificial magnetic fields, one finds that the assumption leading to the Thomas-Fermi regime becomes incorrect. We assumed that the size ξ of the vortex cores was small compared to the distance between vortices, so that the corresponding density hole could be neglected and the true wave function of the condensate replaced by a smoothed wave function. Now, the healing length grows indefinitely as the rotation frequency Ω approaches ω . Indeed, the spatial density at the center of the condensate varies with the effective trap frequency $(\omega^2 - \Omega^2)^{1/2}$ as [cf. VII.30]:

$$\Omega \rightarrow \omega : \quad \rho(0) \propto (\omega^2 - \Omega^2)^{1/2} \Rightarrow \xi \propto (\omega^2 - \Omega^2)^{-1/4}. \quad (\text{VII.63})$$

The interpretation of this result is simple: as the rotation speed is increased, the centrifugal force increasingly deforms the gas, whose size diverges at

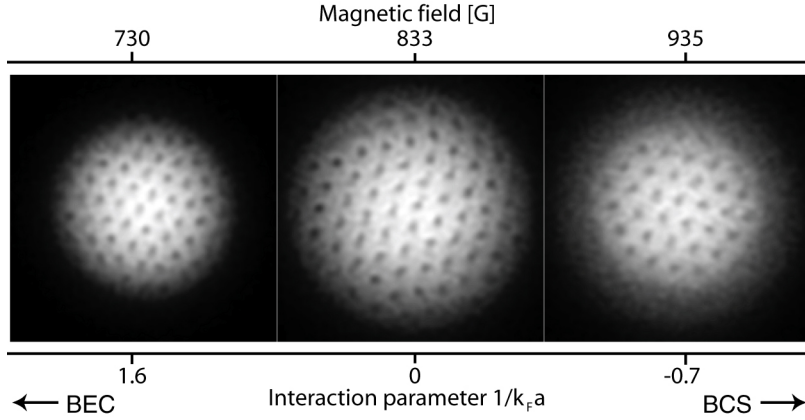


Figure VII.9. Observation by Zwierlein, Abo-Shaeer, et al. (2005) of a vortex lattice for a rotating fermionic superfluid of ${}^6\text{Li}$ atoms. A Feshbach resonance is used to work near the unitary limit $a = \infty$. On the right, the gas is in a BCS regime; on the left, in a molecular-condensate regime. This figure was obtained with 2 million ${}^6\text{Li}$ atoms in a 3D trap corresponding to $\omega/2\pi = 57\text{ Hz}$, for a rotation frequency $\Omega/2\pi = 45\text{ Hz}$.

the limit $\Omega = \omega$. Since the gas becomes more and more dilute, the particles interact less and less, the chemical potential decreases accordingly, and the healing length $\xi \propto \mu^{-1/2}$ increases.

Moreover, in this limit of rapid rotation $\Omega \approx \omega$, Feynman's argument leads to a vortex density

$$\rho_v = \frac{M\omega}{\pi\hbar} = \frac{1}{\pi a_\perp^2}, \quad (\text{VII.64})$$

i.e., a distance between vortices $\sim a_\perp$. There thus exists a rotation frequency at which ξ becomes larger than $a_\perp$:

$$\frac{\Omega}{\omega} \gtrsim 1 - \frac{1}{Ng} \quad (\text{VII.65})$$

Beyond this frequency, our initial assumption is no longer valid, and the description of the condensate's wave function is notably modified. To properly characterize the regime likely to appear, we will first return to the single-particle states and the notion of the Landau level.

3-1 Reminder: Landau levels in the symmetric gauge

We encountered in chapter 2 the eigenstates of the single-particle Hamiltonian $\hat{H}_0$ corresponding to a two-dimensional isotropic oscillator. Let us recall the main results:

- The energy levels are labeled by an integer $n_0 \geq 0$ and are written:

$$E_{n_0} = (n_0 + 1)\hbar\omega. \quad (\text{VII.66})$$

- Each level has degeneracy $n_0 + 1$: the ground level $n_0 = 0$ is non-degenerate, the level $n_0 = 1$ is doubly degenerate, etc. These single-particle levels are shown in figure VII.10a.
- Since the single-particle Hamiltonian commutes with the z component of the angular-momentum operator $\hat{L}_z$, one can find a joint eigenbasis for the two operators. Recall that the eigenfunctions of $\hat{L}_z = -i\hbar\partial_\varphi$ have angular dependence $e^{im\varphi}$, and that the corresponding eigenvalue for $\hat{L}_z$ is $m\hbar$, where m is a positive, negative, or zero integer. The states of the joint eigenbasis of $\hat{H}$ and $\hat{L}_z$ are thus labeled by the two quantum numbers n_0 and m , and denoted $|n_0, m\rangle$.
- The $n_0 + 1$ possible values of L_z in the eigenspace of $\hat{H}$ with energy $(n_0 + 1)\hbar\omega$ are the $m\hbar$ with:

$$m \in \{-n_0, -n_0 + 2, \dots, n_0 - 2, n_0\}, \quad \text{i.e., } |m| \leq n_0, \quad n_0 - m \text{ even.} \quad (\text{VII.67})$$

In particular, the states

$$\phi_m(r, \varphi) \propto r^m e^{im\varphi} e^{-r^2/2a_\perp^2}, \quad (\text{VII.68})$$

are eigenstates of the single-particle Hamiltonian with energy $(m + 1)\hbar\omega$, and thus correspond to the states $|n_0 = m, m\rangle$.

Let us now perform the passage to the rotating frame $\hat{H}_0 \rightarrow \hat{H} = \hat{H}_0 - \Omega\hat{L}_z$. The search for the eigenstates of this single-particle Hamiltonian is modeled on what we saw in chapter 2 for the Landau problem in the symmetric gauge. More precisely, the states $|n_0, m\rangle$ remain eigenstates of $\hat{H}$ with eigenvalues (figure VII.10b-c)

$$E_{n_0, m} = (n_0 + 1)\hbar\omega - m\hbar\Omega, \quad |m| \leq n_0, \quad n_0 - m \text{ even.} \quad (\text{VII.69})$$

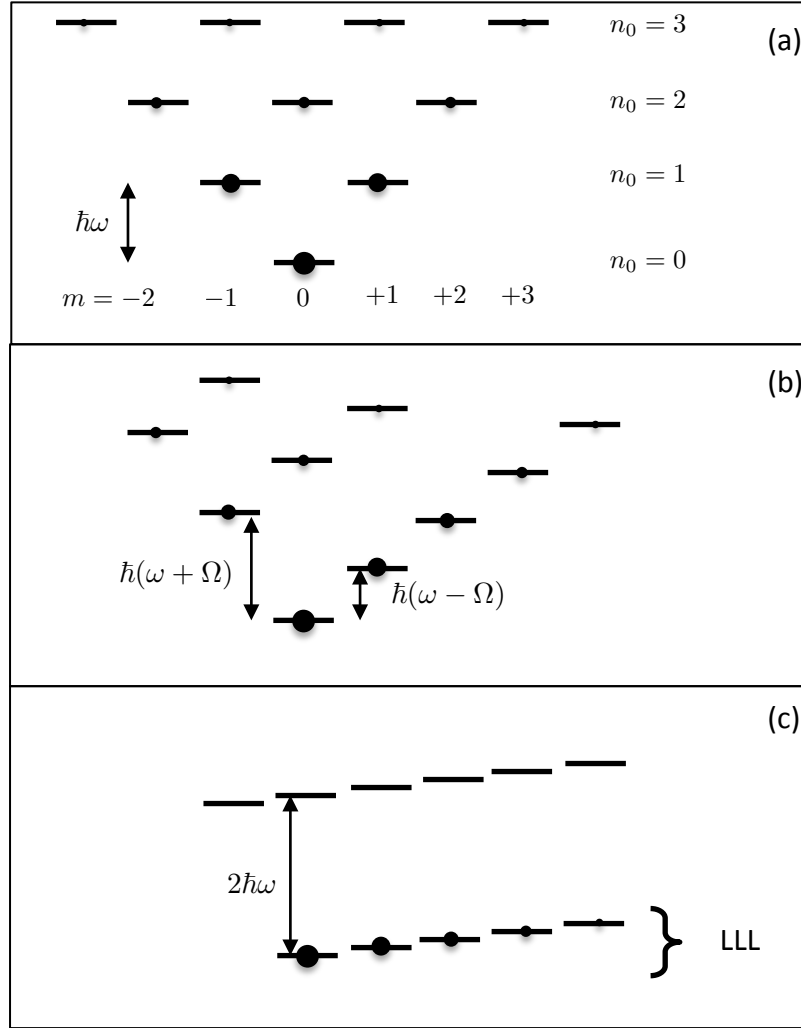


Figure VII.10. (a) Levels of the two-dimensional isotropic harmonic oscillator. The black dots correspond (qualitatively) to the populations of a mean-field state whose chemical potential μ is a few $\hbar\omega$. (b-c): Energy levels of the Hamiltonian of a 2D isotropic harmonic oscillator in the rotating frame, for increasing values of the rotation frequency Ω . (b): $\Omega = 0.4\omega$, (c) $\Omega = 0.9\omega$. For (c), the populations of a state localized in the lowest Landau level (LLL) are shown with black dots.

In particular, the states ϕ_m are eigenstates of $\hat{H}$ with eigenvalues

$$\text{States } \phi_m \text{ of (VII.68): } E_{n_0=m, m} = m\hbar(\omega - \Omega) + \hbar\omega. \quad (\text{VII.70})$$

When the rotation frequency Ω equals the trapping frequency ω , the states group into degenerate multiplets, corresponding to the energies

$$\Omega = \omega : \quad E_n = (2n + 1)\hbar\omega, \quad n = \frac{n_0 - m}{2} \in \mathbb{N}, \quad (\text{VII.71})$$

and we recover the equally spaced Landau levels, with spacing $\omega_c = 2\omega$ between two consecutive levels. The lowest Landau level, $n = 0$, is spanned by the set of ϕ_m .

If we restrict ourselves to Ω close to ω , but still slightly smaller to avoid the centrifugal explosion, the strict degeneracy of the Landau levels is lifted, but the single-particle spectrum continues to group into multiplets, with two clearly distinct energy scales (cf. figure VII.10c):

- a small energy scale $\hbar(\omega - \Omega)$, which corresponds to the distance between two states $|n_0, m\rangle$ and $|n_0 + 1, m_0 + 1\rangle$ of the same multiplet,
- a large energy scale $\hbar(\omega + \Omega) \approx \hbar\omega_c$ corresponding to the distance between the lowest levels of two adjacent multiplets.

3-2 LLL functions and vortices

Once the eigenstates of the single-particle Hamiltonian are identified, one can return to the condition

$$\xi \geq a_\perp \Leftrightarrow \mu \leq \hbar\omega \quad (\text{VII.72})$$

which marks the end of the validity of the traditional Thomas–Fermi approximation. It is clear from the energy diagram in figure VII.10c that the inequality $\mu \leq \hbar\omega$ means that only the LLL states are significantly populated in this regime. Entering this regime corresponds to a very particular form of the ground state of the system, which is now of the form:

$$\phi(\mathbf{r}) = \sum_m \alpha_m \phi_m(\mathbf{r}). \quad (\text{VII.73})$$

Each function $\phi_m(\mathbf{r})$ being written

$$\phi_m(\mathbf{r}) \propto r^m e^{im\varphi} e^{-r^2/2a_\perp^2} = u^m e^{-r^2/2a_\perp^2} \quad \text{with} \quad u = x + iy, \quad (\text{VII.74})$$

an LLL function is of the type

$$\phi(\mathbf{r}) = P(u) e^{-r^2/2a_\perp^2}, \quad (\text{VII.75})$$

where $P(u)$ is a polynomial or an analytic function of the complex variable u . For simplicity, let us restrict ourselves to the case of polynomials: exploring the LLL amounts to choosing an (arbitrary) degree $m_{\max}$ for this polynomial and its $m_{\max} + 1$ coefficients, or, equivalently, choosing the $m_{\max}$ zeros $u_1, u_2, \dots, u_{m_{\max}}$ of the polynomial and writing $\phi(\mathbf{r})$ in the form

$$\phi(\mathbf{r}) \propto \prod_{m=1}^{m_{\max}} (u - u_m) e^{-r^2/2a_\perp^2}. \quad (\text{VII.76})$$

The function $\phi(\mathbf{r})$ has a vortex structure around each zero u_m , with a positive phase winding of $+2\pi$. Indeed, in the immediate vicinity of a given zero, for example u_1 , the variation of $\phi(\mathbf{r})$ is $\phi(\mathbf{r}) \sim C(u - u_1)$, where C is a constant, which indeed corresponds to a phase winding of $+2\pi$ when going around u_1 in the positive direction.

In the LLL, there is a one-to-one correspondence between the position of the vortices and the wave function itself (Ho 2001). Starting from the expression (VII.76), the atomic density ρ and the vortex density ρ_v

$$\rho(\mathbf{r}) = |\phi(\mathbf{r})|^2, \quad \rho_v(\mathbf{r}) = \sum_{m=1}^{m_{\max}} \delta(\mathbf{r} - \mathbf{r}_m), \quad (\text{VII.77})$$

where $\mathbf{r}_m$ is the point with coordinates $x = \text{Re}(u_m)$, $y = \text{Im}(u_m)$, are related by

$$\Delta \{\ln[\rho(\mathbf{r})]\} = 4\pi\rho_v(\mathbf{r}) - \frac{4}{a_\perp^2}. \quad (\text{VII.78})$$

This relation follows directly from the two-dimensional mathematical identity

$$\Delta [\ln|\mathbf{r} - \mathbf{r}_0|] = 2\pi \delta(\mathbf{r} - \mathbf{r}_0). \quad (\text{VII.79})$$

Let us note an interesting limiting case of relation (VII.78). If one takes a uniform vortex density in the plane, equal to $1/(\pi a_\perp^2)$, then the spatial

density is also uniform. This case corresponds to exactly degenerate Landau levels, where the gas spreads over the entire plane. This situation is equivalent to the limit obtained in (VII.64) by taking exactly $\Omega = \omega$.

Starting from (VII.78), Ho (2001) made the *ad hoc* assumption of a uniform vortex density, but slightly lower than $1/(\pi a_\perp^2)$. One deduces from (VII.78) that $\ln[\rho(\mathbf{r})]$ varies quadratically with r , and hence that $\rho(\mathbf{r})$ is a Gaussian, after averaging over the holes caused by the vortex distribution. More precise analyses [see for example Cooper, Komineas, et al. (2004), Watanabe, Baym, et al. (2004), Aftalion, Blanc, et al. (2005), and Matveenko, Kovrizhin, et al. (2009)] have refined and somewhat corrected this hypothesis of Ho (2001). In fact, the density profile that minimizes the energy is (approximately) an inverted parabola, not a Gaussian. This inverted parabola $\rho(\mathbf{r}) \propto R^2 - r^2$ vanishes beyond the radius:

$$R = a_\perp \left(\frac{2}{\pi} \frac{bNg}{1 - \Omega/\omega} \right)^{1/4}. \quad (\text{VII.80})$$

We thus recover the form of the Thomas–Fermi solution (VII.61), with an additional multiplicative parameter denoted b . This number $b \approx 1.1596$ is the *Abrikosov parameter*. It describes the fact that, in the presence of a vortex lattice with density $\rho_v \sim 1/a_\perp^2$, the atoms interact with one another slightly more than if their density were uniform. The holes created by the vortices must be compensated by regions of higher density, and there is thus an additional price to pay in terms of interaction energy in the presence of a vortex lattice.

Since vortex density and atomic density are related by (VII.78), the fact that the atomic density is not Gaussian implies that the vortex density is not uniform. This is indeed what results from minimizing the energy functional. We have shown in figure VII.11 the vortex distribution and the corresponding atomic distribution. One sees that at the edges of the cloud, the vortex lattice loses its triangular character and becomes distorted. More precisely, the vortex density drops, which has the effect of increasing the curvature of $\ln[\rho(\mathbf{r})]$, and thus of causing this density to drop at the edges of the cloud faster than for a Gaussian function.

Let us finally note that one can easily estimate the number of functions ϕ_m that are significantly populated in the ground state. Since a function ϕ_m is peaked at $r \sim \sqrt{m} a_\perp$, the radius R of the cloud provides a measure

of the index $m_{\max}$ of the populated LLL functions. This number $m_{\max}$ then gives the number of zeros of the polynomial characterizing the state of the system, and hence the number of vortices inside the cloud:

$$N_v = m_{\max} \approx \frac{R^2}{a_{\perp}^2} \approx \left(\frac{bgN}{1 - \Omega/\omega} \right)^{1/2} \quad (\text{VII.81})$$

where coefficients of order unity have been neglected. One can compute, within this same approximation, the average angular momentum per particle

$$\langle L \rangle \approx \frac{m_{\max}}{3} \hbar. \quad (\text{VII.82})$$

The number of populated states thus grows indefinitely as the rotation frequency Ω approaches ω . When this number $m_{\max}$ becomes of the order of the number of particles N , the mean-field approximation is no longer valid, and one enters the strongly correlated regime; we will address this regime in part 4. The estimate (VII.81) predicts that this number of states becomes comparable to N when $\Omega \sim \omega(1 - g/N)$. Using the critical frequency for the first vortex (VII.43) and the condition for entering the LLL (VII.65), one can thus separate the rotation frequencies into different regimes

$$\frac{\Omega_c}{\omega} < \frac{\Omega}{\omega} < 1 - \frac{1}{gN} : \text{ vortices of size } \xi \ll a_{\perp}, \quad (\text{VII.83})$$

$$1 - \frac{1}{gN} < \frac{\Omega}{\omega} < 1 - \frac{g}{N} : \text{ LLL, vortices of size } a_{\perp}, \quad (\text{VII.84})$$

$$1 - \frac{g}{N} < \frac{\Omega}{\omega} : \text{ outside the mean-field regime. } (\text{VII.85})$$

3-3 Experiments in the LLL

We already described in chapter 4 the method of setting into rotation by selective evaporation (*evaporative spinup*) developed by the Boulder group to reach very high rotation speeds $\Omega = 0.993\omega$. This made it possible to reach the LLL regime, which manifests itself experimentally by a significant increase in the size of the vortex cores. We have shown in figure VII.12 the fraction of the area occupied by the vortex cores as a function of the rotation speed in the Boulder experiment (Schweikhard, Coddington, et al.

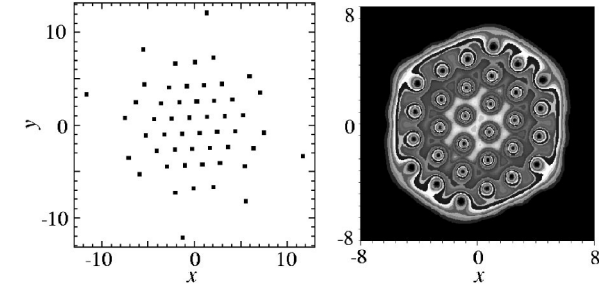


Figure VII.11. Example of the distribution for the vortices (left) and for the atomic density (right) in the LLL. This figure is taken from Aftalion, Blanc, et al. (2005) and corresponds to $Ng/(1 - \Omega/\omega) = 3000$. Lengths are given in units of $a_{\perp}$.

2004; Coddington, Haljan, et al. 2004). For low rotation speeds, this fraction is small compared to 1, as expected since $\xi \ll a_{\perp}$. For high rotation speeds, this fraction becomes very significant ($\sim 20\%$), since the core size and the vortex spacing are then comparable, of order $a_{\perp}$. The parameters of this experiment correspond to a coupling constant $g \approx 5 \cdot 10^{-3}$ with a number of atoms of order 10^5 .

Another method for approaching the LLL, studied at the ENS, consists in adding a quartic potential ηr^4 to the harmonic confining potential $m\omega^2 r^2/2$. This makes it possible to choose rotation frequencies Ω close to or equal to ω without risking losing the gas because of the centrifugal force (Bretin, Stock, et al. 2004).

4 Beyond mean field

To motivate this last paragraph, let us return to the validity criterion of the mean-field treatment we have used up to now. For a mean-field state to be a good approximation of the N -body ground state, the number of significantly populated eigenstates ϕ_m of the single-particle Hamiltonian must remain small compared to the number of atoms N . If this is not the case, one can generally construct states exhibiting strong correlations between particles, which lead to a notable lowering of the energy compared to the

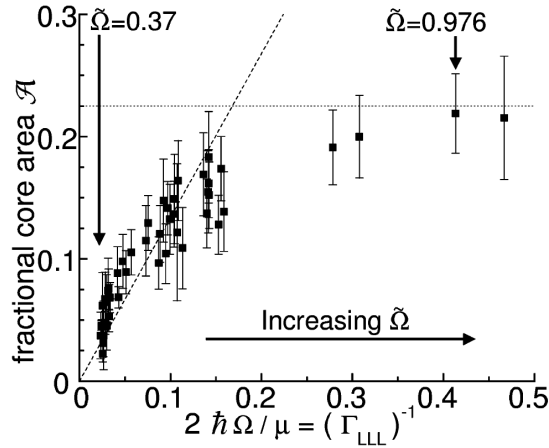


Figure VII.12. Fraction of space occupied by the vortex cores as a function of the rotation speed Ω . Entry into the LLL regime results in a fraction that is not negligible compared to unity [figure taken from Schweikhard, Coddington, et al. (2004)].

mean-field result.

For the problem of the rotating Bose gas, we already indicated in (VII.85) that entry into the correlated regime corresponds to $\Omega \gtrsim \omega(1 - g/N)$. The analysis of this ultra-fast rotation regime shows strong analogies with that of the fractional quantum Hall effect, in which one is interested in the correlated states of a gas of electrons with Coulomb interactions in the presence of a strong magnetic field. In both cases, the interesting states are generally confined to the LLL and possess a remarkable property, incompressibility: the ground state is separated from all the excited states of the system by a gap that is independent of the system size and is proportional to the interaction strength g .

4-1 Ground state with N particles in the LLL

We saw above that the single-particle states are of the form $\phi(\mathbf{r}) = P(u) \exp(-r^2/2a_\perp^2)$, with $u = x + iy$ and $P(u)$ a polynomial in the vari-

able u . For a system of spinless bosons, a general N -body state is thus of the form

$$\Phi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = \mathcal{P}(u_1, u_2, \dots, u_N) \exp(-\sum_i r_i^2/2a_\perp^2), \quad (\text{VII.86})$$

where $\mathcal{P}(u_1, u_2, \dots, u_N)$ is a symmetric polynomial of the N complex variables u_i . Finding the eigenstates of the N -body Hamiltonian is generally done by an exact diagonalization method for not-too-large numbers of particles (of order ten).

The N -body Hamiltonian commutes with the total angular-momentum operator $\hat{L}_z$, and one can thus look for a joint eigenbasis of the two operators. We will be interested here in the ground state of the Hamiltonian for a given total angular momentum $\mathcal{L}$. Now, the angular momentum of a single-particle LLL function, $u^m \exp(-r^2/2a_\perp^2)$, is directly given by the value of the exponent m . For the N -particle system, we will thus look for the polynomial $\mathcal{P}$ giving the ground state at fixed $\mathcal{L}$ as a sum of monomials all having the same total degree:

$$u_1^{\alpha_1} \dots u_N^{\alpha_N} \quad \text{with} \quad \sum_i \alpha_i = \mathcal{L}. \quad (\text{VII.87})$$

For example, for a system of 3 particles and a total angular momentum $\mathcal{L} = 3$, one seeks the polynomial $\mathcal{P}$ corresponding to the ground state in the form

$$\begin{aligned} \mathcal{P}(u_1, u_2, u_3) &= \alpha u_1 u_2 u_3 \\ &+ \beta [(u_1^2(u_2 + u_3) + u_2^2(u_1 + u_3) + u_3^2(u_1 + u_2))] \\ &+ \gamma (u_1^3 + u_2^3 + u_3^3). \end{aligned}$$

The first term of this sum, proportional to $u_1 u_2 u_3$, corresponds to a *mean-field*-type state, more precisely a centered vortex, since the three particles are in the same state $\phi_1(\mathbf{r})$. The other terms of the sum correspond to correlated states of the three particles.

The procedure for finding the N -body ground state is thus the following:

⁴In this part, the total angular momentum is expressed in units of $\hbar$. The quantity $\mathcal{L}$ is thus a non-negative integer.

- One chooses a total angular momentum $\mathcal{L}$, corresponding to a subspace of the LLL for which the symmetric polynomials are built from the monomials (VII.87).
- One considers, within this subspace, the ground state of the Hamiltonian

$$\hat{H} = \sum_i \left(\frac{\hat{p}_i^2}{2M} + \frac{1}{2} M \omega^2 \hat{r}_i^2 \right) + \frac{\hbar^2}{M} g \sum_{i < j} \delta^{(2)}(\mathbf{r}_i - \mathbf{r}_j). \quad (\text{VII.88})$$

Now, the one-body part of this Hamiltonian gives the same result $(\mathcal{L} + N)\hbar\omega$ for all the states of the subspace considered. One thus reduces the problem to the diagonalization of the interaction Hamiltonian

$$\hat{H}_{\text{int.}} = \frac{\hbar^2}{M} g \sum_{i < j} \delta^{(2)}(\mathbf{r}_i - \mathbf{r}_j), \quad (\text{VII.89})$$

within this subspace.

- Once the interaction energy is known, one obtains the total energy in the rotating frame by adding the term $(\mathcal{L} + N)\hbar\omega - \Omega\mathcal{L}$ to the eigenvalue found for $\hat{H}_{\text{int.}}$.

4-2 A few remarkable states

The description of all the correlated states that have been identified with this exact diagonalization procedure goes far beyond the scope of this course, and we refer the reader to the very comprehensive review article by Cooper (2008) for a detailed presentation. We will describe here a few remarkable phases that have been identified when the ratio $\mathcal{L}/N$ is varied.

Melting of the vortex lattice. Starting from a regime well described by mean field, let us increase the rotation speed of the gas; the first deviation expected from this mean-field regime is the melting of the vortex lattice due to quantum fluctuations. This melting was predicted by Cooper, Wilkin, et al. (2001) using an exact diagonalization technique, for filling factors $\nu = N/N_v$ between 6 and 10. This value can be recovered by computing the amplitude Δ of the quantum fluctuations of the vortex positions

and applying the heuristic Lindemann criterion. These fluctuations are expected to destabilize the lattice when their amplitude becomes of order a tenth of the vortex spacing $\rho_v^{-1/2}$ (Sinova, Hanna, et al. 2002).

The Laughlin state. This is a remarkable, strongly correlated state, which is an exact eigenstate of the N -body Hamiltonian. The symmetric polynomial corresponding to this state is written

$$\mathcal{P}_{\text{Lau.}}(u_1, u_2, \dots, u_N) = \prod_{i < j} (u_i - u_j)^2, \quad (\text{VII.90})$$

of total degree $\mathcal{L} = N(N - 1)$ and spanning all the single-particle LLL functions from $m = 0$ to $m_{\text{max}} = 2(N - 1)$, i.e., a filling factor $\nu = 1/2$. The action of $\hat{H}_{\text{int}}$ on this state gives strictly 0, since $\hat{H}_{\text{int}}$ corresponds to a contact interaction, whereas the Laughlin state corresponds to a strictly zero probability of finding two particles at the same place.

The Laughlin state is incompressible, and the gap to the first excited state of the same angular momentum is $\approx 0.1 g \hbar\omega$ (Regnault & Jolicoeur 2003; Regnault & Jolicoeur 2004). The Laughlin state is characterized by a nearly uniform distribution of particles over a disk of radius $a_{\perp} \sqrt{2N}$.

Descendants of the Laughlin state. For angular momenta larger than the one giving rise to the Laughlin state $\mathcal{L} = N(N - 1)$, any state of the type

$$\mathcal{P}(u_1, \dots, u_N) = \mathcal{P}_{\text{Lau.}}(u_1, \dots, u_N) Q(u_1, \dots, u_N) \quad (\text{VII.91})$$

where Q is an arbitrary symmetric polynomial in $u_1, \dots, u_N$, will likewise be a state of strictly zero interaction energy. The physical interpretation of these states depends on the degree of Q . For a degree of order 1, these are edge excitations of the Laughlin state; by choosing a degree equal to N , one can construct quasi-holes at a given point $U_0 = X_0 + iY_0$ by taking (Paredes, Fedichev, et al. 2001; Paredes, Zoller, et al. 2002)

$$Q(u_1, \dots, u_N) = \prod_j (u_j - U_0) \quad (\text{VII.92})$$

The Moore-Read state. For even N , Cooper, Wilkin, et al. (2001) identified the state of angular momentum $\mathcal{L} = N(N-2)/2$, also called the *Pfaffian*:

$$\mathcal{P}(u_1, \dots, u_N) = \mathcal{S} \left[\prod_{i < j \leq N/2} (u_i - u_j)^2 \prod_{N/2 < l < n} (u_l - u_n)^2 \right] \quad (\text{VII.93})$$

as being, to a very good approximation, an eigenstate of $\hat{H}_{\text{int}}$ ($\mathcal{S}$ denotes the symmetrization operator). More precisely, the overlap of the state (VII.93) with the true eigenstate determined numerically for this value of $\mathcal{L}$ exceeds 0.88 as long as $N \leq 14$ (Chang, Regnault, et al. 2004). This state decomposes over single-particle states⁵ ranging from $m = 0$ to $m = N-2$, corresponding to a filling factor $\nu = 1$. This state is likewise separated by a gap $\sim 0.05 g \hbar \omega$ from the excited states of the same $\mathcal{L}$ (Chang, Regnault, et al. 2004). Note that for the approximate state (VII.93), the probability of having three particles at the same point is zero.

4-3 Possible detection schemes

We conclude this paragraph with a few methods that have been proposed for experimentally detecting these strongly correlated states. Let us first note that the observation of these states will probably be limited to small numbers of atoms if one uses the method of rotating a harmonic trap. Indeed relation (VII.85), defining the entry into the correlated regime, combined with an interaction strength $g \sim 1$ and a realistic rotation frequency $\Omega \approx 0.99 \omega$, gives a maximum number of atoms of order 100.

A first signature of *Laughlin-like* or *Moore-Read-like* states would be to see a reduction in losses due to inelastic collisions. Indeed, the probability of having 3 particles at the same place is zero for these states (whereas it is not zero for mean-field states). Another approach is related to the incompressibility of these states, which gives rise to a “wedding-cake”-shaped

density profile in a harmonic trap, each plateau of the wedding cake corresponding to a different incompressible state. A third possibility is to use a detection system with very good spatial resolution to detect the atoms of the gas one by one (possibly after a time of flight), and thus reconstruct the spatial correlation functions that make it possible to characterize the initial state.

Let us finally mention a more ambitious proposal (Paredes, Fedichev, et al. 2001). This involves studying the possibility of creating anyons in a gas described by a *fractional-quantum-Hall-effect*-type state (Wilczek 1982); starting from a Laughlin state, one would dig a hole in the gas using a tightly focused laser beam, thereby producing a state of the type (VII.92). By adiabatically moving this hole through the gas, the system should acquire a geometric phase that could then be measured by an interferometry experiment. Measuring the accumulated phase should reveal the anyonic character (neither boson nor fermion) of the quasiparticle thus created by the laser beam.

⁵For example, for $N = 4$, one has

$$\mathcal{P}(u_1, \dots, u_4) = (u_1 - u_2)^2(u_3 - u_4)^2 + (u_1 - u_3)^2(u_2 - u_4)^2 + (u_1 - u_4)^2(u_2 - u_3)^2.$$

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