Basic Concepts and some current Directions in Ultracold Gases

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These are notes on a series of lectures on many-body phenomena in ultracold gases at the Collège de France in the Fall of 2021. Their main focus are Bose systems, for a review of strongly interacting Fermi gases see the Varenna Lectures 2014, accesssible via arXiv:1608.00457. As an introductory comment, I quote from the preface of the two volume book on 'Statistical Field Theory' by C. Itzykson and J.-M. Drouffe who remark:

' A book might give the illusion, especially to students, that some knowledge has become definitive and that the authors understand every part of it. This is a completely false view. No one can really fully master even his own subject, and this is luckily a source of progress.'

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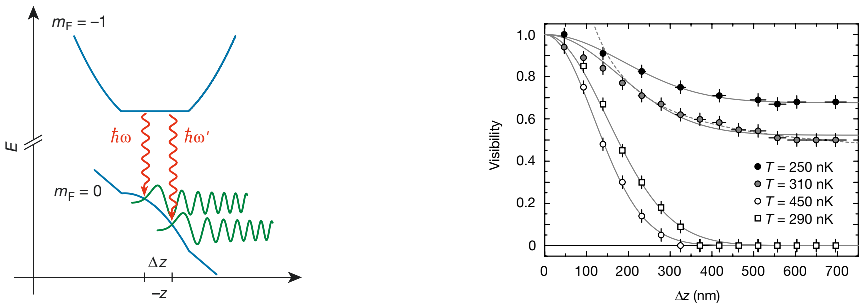

FIG. 1: Schematic setup of out-coupling two atom beams from a trapped BEC via RF-transitions into an untrapped hyperfine state $m_F = 0$. The visibility of the resulting interference fringes as a function of separation is shown on the right. Below the BEC transition temperature of $T_c \simeq 400\,\mathrm{nK}$, the visibility approaches a constant for separations exceeding about ten average interparticle distances, thus providing direct evidence for the presence of off-diagonal long range order. The Figures are taken from Bloch et al. [2].

I. SUPERFLUIDITY IN GASES AND LIQUIDS

Off-diagonal long range order and Widom particle insertion A precise definition of Bose-Einstein condensation (BEC) in an interacting system has been given by Penrose [1]. It is based on the concept of off-diagonal long range order (ODLRO) which states that the off-diagonal elements

$$\lim_{|\boldsymbol{x}-\boldsymbol{x}'|\to\infty} \rho_1(\boldsymbol{x},\boldsymbol{x}') = \lim_{|\boldsymbol{x}-\boldsymbol{x}'|\to\infty} \langle\hat{\psi}^\dagger(\boldsymbol{x})\hat{\psi}(\boldsymbol{x}')\rangle = n_0 \neq 0 \tag{1}$$

of the one-particle density operator $\hat{\rho}_1$ approach a finite constant at arbitrary large separation. The limit defines a condensate density n_0 which is the square of the order parameter for BEC in the interacting system. Physically, the condition (1) reflects the presence of long range phase coherence: states in which one particle is removed either at $\boldsymbol{x}$ or at a distant position $\boldsymbol{x}'$ have a finite overlap for arbitrary large separation. Experimentally, this property has first been observed in the context of ultracold gases by Bloch et al. [2]. As shown in Fig. 1, the visibility in the interference from two beams outcoupled at separate points of a trapped BEC decreases to zero as a function of separation above the critical temperature while it stays finite below the transition.

In the following, we want to ask what are necessary and sufficient conditions in the ground state many-body wave function for the existence of ODLRO. Specifically, we consider a generic non-relativistic Hamiltonian with pure two-body interactions. The associated first quantized Hamiltonian

$$\hat{H}_N = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\nabla_i^2 + \sum_{1\le i<j\le N} V(\boldsymbol{x}_i-\boldsymbol{x}_j) \tag{2}$$

gives rise to a proper thermodynamics with an extensive free energy and a positive compressibility provided the interaction obeys

$$\sum_{1\le i<j\le N} V(\boldsymbol{x}_i-\boldsymbol{x}_j) > -B\cdot N\,. \tag{3}$$

Here, B is a positive constant independent of the specific state. As shown by Fisher [3], a sufficient condition for the validity of Eq. (3) is that the two-body potential $V(r) \geq -\epsilon$ has a finite lower bound, decays faster than $1/r^3$ at large distances and increases more rapidly than $1/r^3$ for separations smaller than a short range scale σ. More specifically, we consider interactions with an asymptotic van der Waals tail $V(r\to\infty) = -C_6/r^6$. Apart from σ, they are characterized by the van der Waals length $\ell_{\mathrm{vdW}} = (mC_6/\hbar^2)^{1/4}/2$ as a second length scale which is determined solely by the asymptotic part of the interaction. A standard example is the Lennard-Jones potential $V(r) = 4\epsilon\,[(\sigma/r)^{12} - (\sigma/r)^6]$ where the short distance scale σ and the depth ϵ are connected with the strength of the van der Waals tail via $C_6 = 4\epsilon\,\sigma^6$. Independent of the precise form of $V(r)$, the equilibrium free energy $F(N) = f\,N + \ldots$ for the class of potentials obeying (3) scales linearly with the particle number. At zero temperature, the generic ground state even in the limit of vanishing pressure is a solid, where both the particle statistics and zero point fluctuations play only a minor role. A measure for their strength is provided by the parameter

$$\Lambda_{\mathrm{dB}} = \frac{\hbar}{\sigma\sqrt{m\epsilon}} \underset{\mathrm{vdW}}{\longrightarrow} \frac{1}{2}\left(\frac{\sigma}{\ell_{\mathrm{vdW}}}\right)^2 \tag{4}$$

introduced by De Boer [4], which is the square root of the ratio between the zero point energy on the scale σ and the depth ϵ of the attractive part of the potential. From numerical studies, Nosanow et al. [5] found that for bosons the crystalline solid ground

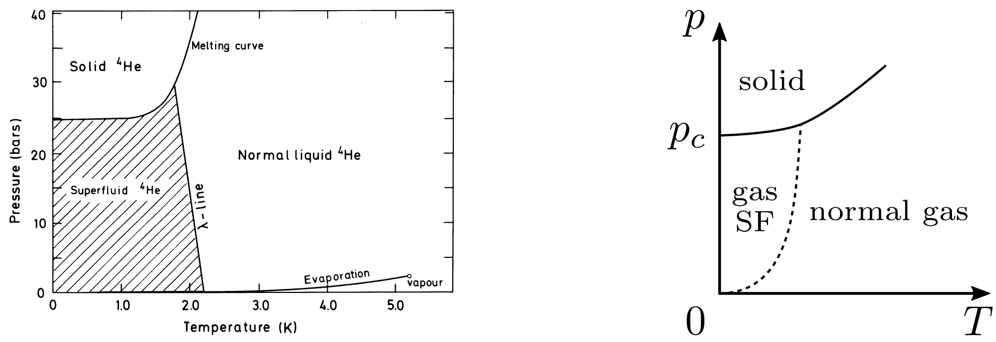

FIG. 2: The Figure on the left shows the phase diagram of ^{4}He, whose ground state is a superfluid liquid below a critical pressure $p_c \simeq 25$ bar. On the right, a qualitative phase diagram is shown for a Bose system in the regime $\Lambda_{\rm dB} > \Lambda^c_{\rm dB}$, where the ground state at low pressure is a superfluid gas. The continuous transition from the superfluid to the normal gas asymptotically exhibits a cubic dependence $p(T) \simeq g/\lambda_T^6 \sim g\,T^3$.

state realized for small values of $\Lambda_{\rm dB}$ melts into a liquid at a non-universal critical value $\Lambda_{\rm dB} \simeq 0.37$ via a first order quantum phase transition. Both the solid and the liquid phase have a finite density $\bar{n}$ at vanishing pressure and a negative ground state energy $u(\bar{n})$ per particle. Specifically, for ^{4}He, where $\Lambda_{\rm dB} \simeq 0.42$, precise results for the dimensionless density $\bar{n}\sigma^3 \simeq 0.364$ or the energy per particle $u(\bar{n}) \simeq -0.7\,\epsilon \simeq -k_B \cdot 7\,$K are available by numerical methods [6, 7]. Upon further increasing the strength of the zero point fluctuations, the liquid eventually unbinds into a gas through a continuous quantum phase transition at $\Lambda^c_{\rm dB} \simeq 0.68$. This transition was first studied numerically by Miller et al. [8] and will be discussed in more detail below. The phase diagram at finite temperature beyond $\Lambda^c_{\rm dB}$, which has neither a triple nor a critical point, is sketched in Fig. 2. As a true equilibrium configuration it is realized only for spin polarized hydrogen, where $\Lambda_{\rm dB} \simeq 0.74$ [9]. A gaseous superfluid near vanishing pressure and temperature is also present in ultracold Alkali gases, even though their de Boer parameter is much less than one. This is a result of the fact that in the regime of very low densities $n\ell^3_{\rm vdW} \ll 1$, the short distance length σ can effectively be taken to zero and the liquid or solid equilibrium phases are not reached because states with negative energy are inaccessible kinematically with just two-body collisions. As a result, the Hamiltonian can be truncated to one involving only states in the continuum. For positive two-body scattering length a, a gaseous state then forms a stable equilibrium configuration. In the following, we will show that the ground state of a Bose system always exhibits BEC or the related phenomenon of superfluidity provided it is a homogeneous fluid, i.e. either a liquid or a gas. For solids with broken translation invariance, superfluidity may still be present, however generically this requires a finite defect density in their ground state.

It was observed by Feynman [10] that the many-body ground state wave function $\psi(\boldsymbol{x}_1, \boldsymbol{x}_2 \ldots \boldsymbol{x}_N)$ of a Bose system with a permutation symmetric and real Hamiltonian of the form (2) has no nodes. In fact, this is a special case of a more general theorem which states that the lowest energy in an unconstrained minimization of $\langle\psi|\hat{H}|\psi\rangle$ is realized for a positive and symmetric wave function ('minimizers are bosonic') [1]. The theorem relies on the observation that $|\psi|$ gives the same energy as ψ itself and that in a decomposition $\psi = \psi_s + \psi_r$ into a permutation symmetric part ψ_s and a remainder, the cross terms in $\langle\psi|\hat{H}|\psi\rangle$ vanish (for a rigorous proof see Lieb and Seiringer [12], chapter 3.2.4). In order to deal with Bose fluids with strong interactions as in ^{4}He, Feynman and later Penrose and Onsager [13] suggested to express the symmetric and positive many-body wave function

$$\psi(\boldsymbol{x}_1, \boldsymbol{x}_2 \ldots \boldsymbol{x}_N) = [p_{\rm cl}(\boldsymbol{x}_1, \boldsymbol{x}_2 \ldots \boldsymbol{x}_N)]^{1/2} = \frac{1}{\sqrt{Q_N}} \exp\{-\tilde{V}_N(\boldsymbol{x}_1, \boldsymbol{x}_2 \ldots \boldsymbol{x}_N)/2\} \quad (?) \tag{5}$$

in terms of the square root of a N-body probability density of a classical fluid at some finite effective temperature. The normalization is provided by the classical configuration integral $Q_N = \int d1 \ldots dN \, \exp\{-\tilde{V}_N(1 \ldots N)\}$. In principle, such a representation is always possible by defining the dimensionless effective potential $\tilde{V}_N(1 \ldots N)$ of the classical reference system such that the square of (5) is obeyed as an identity. This is used e.g. in Laughlin's plasma analogy for incompressible states in the lowest Landau level, connecting the square of Ansatz wavefunctions to a 2d Coulomb gas with logarithmic interactions [14]. In the present context, however, the idea is useful only if $\tilde{V}_N$ is similar to the underlying microscopic interaction in the quantum many-body problem. As will be shown below, this is actually impossible for any compressible Bose fluid. An assumption which is often made in addition is that the classical reference system can be described by a sum $\tilde{V}_N(1 \ldots N) = \sum_{i<j} \tilde{v}(r_{ij})$ involving a

[1] The theorem also implies that the two-electron ground state of a spin-independent Hamiltonian is always a singlet, see problem 2 in Ref. [11] p. 689 .

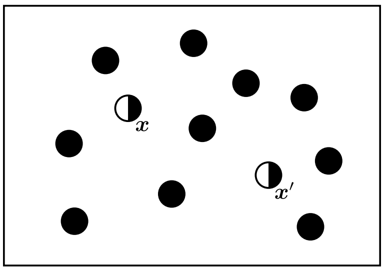

FIG. 3: Widom particle insertion: Two particles at $\boldsymbol{x}$ respectively $\boldsymbol{x}'$ are added to a classical fluid of $N-1$ particles at positions $\boldsymbol{x}_2 \ldots \boldsymbol{x}_N$, represented by full discs. The strength of the interaction of the added particles with those of the fluid is half of that within the fluid itself.

translation and rotation invariant two-body interaction $\tilde{v}(r)$. In this case, the many-body wave function

$$\psi_{\text{Jastrow}}(\boldsymbol{x}_1, \boldsymbol{x}_2 \ldots \boldsymbol{x}_N) = \frac{1}{\sqrt{Q_N}} \exp\left[-\sum_{i<j} \tilde{v}(r_{ij})/2\right] \tag{6}$$

is a product of $N(N-1)/2$ identical two-body wave functions, as introduced by Bijl [15] and Jastrow [16]. For the following considerations, this form is not necessary, however. Indeed, quite generally, the representation (5) implies that the one-particle density matrix of the quantum system

$$\rho_1(\boldsymbol{x}, \boldsymbol{x}') = \frac{N}{Q_N} \int d2 \ldots dN \, \exp\left\{-\left[\tilde{V}_N(\boldsymbol{x}, 2 \ldots N) + \tilde{V}_N(\boldsymbol{x}', 2 \ldots N)\right]/2\right\} \tag{7}$$

can be expressed in terms of a Boltzmann weight of a classical N-particle system where one of the particles is either at a position $\boldsymbol{x}$ or at $\boldsymbol{x}'$. As indicated schematically in Fig. 3, the exponent [2]

$$\frac{1}{2}\left[\tilde{V}_N(\boldsymbol{x}, 2 \ldots N) + \tilde{V}_N(\boldsymbol{x}', 2 \ldots N)\right] = \tilde{V}_{N-1}(2 \ldots N) + \Delta_2 \tilde{W}_{1/2}(\boldsymbol{x}, \boldsymbol{x}') \tag{8}$$

may be separated into a contribution $\tilde{V}_{N-1}(2 \ldots N)$ which accounts for the full interaction energy of an $N-1$-particle system plus an additional term $\Delta_2 \tilde{W}_{1/2}(\boldsymbol{x}, \boldsymbol{x}')$ which describes the change in energy associated with adding two particles at positions $\boldsymbol{x}$ and $\boldsymbol{x}'$ that do not interact among themselves. The subscript $1/2$ indicates that they interact with the $N-1$ particles at positions $\boldsymbol{x}_2, \ldots \boldsymbol{x}_N$ with only half the strength of the potential in the $N-1$-particle system. In a similar manner, the configuration integral for N particles

$$Q_N = Q_{N-1} \int d1 \, \langle \exp\{-\Delta_1 \tilde{W}(\boldsymbol{x}_1)\} \rangle_{N-1} \tag{9}$$

can be expressed in terms of an expectation value of the dimensionless interaction energy $\Delta_1 \tilde{W}(\boldsymbol{x}_1)$ associated with adding a single particle at position $\boldsymbol{x}_1$. Here, the average $\langle \ldots \rangle_{N-1}$ is defined by an integration over the positions $\boldsymbol{x}_2, \ldots \boldsymbol{x}_N$ of an $N-1$ - particle system with Boltzmann weight $\exp\{-\tilde{V}_{N-1}(2 \ldots N)\}$ and a normalization through the associated configuration integral Q_{N-1}. For a homogeneous system, $\langle \exp\{-\Delta_1 \tilde{W}(\boldsymbol{x}_1)\} \rangle_{N-1}$ does not depend on $\boldsymbol{x}_1$, which can be choosen as the reference point for the remaining coordinates $\boldsymbol{x}_2, \ldots \boldsymbol{x}_N$. The integral $\int d1$ then just gives a factor V. Moreover, using standard thermodynamic relations, the ratio $Q_N/Q_{N-1} = V \cdot \exp\{-\tilde{\mu}_{\text{ex}}\}$ can be expressed in terms the excess chemical potential $\tilde{\mu}_{\text{ex}} = \tilde{F}_N - \tilde{F}_{N-1} - \tilde{\mu}_{\text{id}}$ of the fluid in units of the thermal energy. In the theory of classical fluids, these relations go back to Widom [17] and are called the Widom particle insertion method. In fact, the extraction of $\tilde{\mu}_{\text{ex}}$ in this manner is an example of an equality due to Jarzynski [18], which relates the excess chemical potential in equilibrium to the exponential average of the work $\Delta_1 \tilde{W}(0)$ needed to add a single particle at fixed total volume V. Using the decomposition in Eq. (8), the one-particle density matrix

$$\rho_1(\boldsymbol{x}, \boldsymbol{x}') = n \, \frac{\langle \exp\{-\Delta_2 \tilde{W}_{1/2}(\boldsymbol{x}, \boldsymbol{x}')\} \rangle_{N-1}}{\langle \exp\{-\Delta_1 \tilde{W}(0)\} \rangle_{N-1}} \xrightarrow[|\boldsymbol{x}-\boldsymbol{x}'| \to \infty]{} n \, \frac{\langle \exp\{-\Delta_1 \tilde{W}_{1/2}(0)\} \rangle^2}{\langle \exp\{-\Delta_1 \tilde{W}(0)\} \rangle} = n_0 \neq 0 \tag{10}$$

of a homogeneous Bose fluid ground state can be expressed as the ratio of two expectation values in a $N-1$-particle state. For large separation $|\boldsymbol{x} - \boldsymbol{x}'|$, this approaches a finite constant quite generally because inserting two particles at widely separated

[2] Note that both $\Delta_2 \tilde{W}_{1/2}(\boldsymbol{x}, \boldsymbol{x}')$ and $\Delta_1 \tilde{W}(\boldsymbol{x}_1)$ in Eq. (9) depend implicitely also on the coordinates $\boldsymbol{x}_2, \ldots \boldsymbol{x}_N$ but this dependence is suppressed.

positions in a classical fluid with short range interactions is equivalent to two independent single particle additions. A particularly simple situation arises by assuming that the classical reference system is a fluid of hard spheres with diameter σ. In this case, $\Delta_1 \tilde{W}_{1/2}(\boldsymbol{x})|_{\rm HS} \equiv \Delta_1 \tilde{W}(\boldsymbol{x})$ because half of the interaction strength is the same as the full interaction. For the hard sphere fluid, the ratio which determines the condensate fraction in (10) is thus equal to $\exp\{-\tilde{\mu}_{\rm ex}\}$. An analytical expression for the associated excess chemical potential, which only depends on the dimensionless filling fraction η, is obtained within the Carnahan-Starling form of the equation of state of a classical hard sphere fluid which gives [3]

$$\tilde{\mu}_{\rm ex}|_{\rm HS} = \eta\,\frac{8-9\eta+3\eta^2}{(1-\eta)^3} \quad \text{with} \quad \eta = Nv_\sigma/V = \frac{\pi}{6}\,n\sigma^3\,. \tag{11}$$

As shown by Penrose and Onsager [13], this approach can be used to provide an estimate for the condensate fraction in ^{4}He. Taking the known value $\sigma \simeq 2.5$Å of the short distance scale below which the ^{4}He - ^{4}He interaction becomes strongly repulsive as an effective hard sphere diameter, the density of liquid ^{4}He is about 0.28 times that of close packing, which corresponds to an effective dimensionless filling fraction $\eta_{\rm eff}(^4{\rm He}) \simeq 0.2$. Based on Eq. (11) for the associated excess chemical potential, this leads to a condensate fraction $n_0/n = \exp -\tilde{\mu}_{\rm ex} = 0.078$. The assumption that the many-body ground state wave function of a Bose fluid has a representation in the form (5) of an effective classical reference system thus leads to two important conclusions:

(a) Any translation invariant ground state of an interacting Bose system is necessarily a superfluid exhibiting ODLRO, because two-particle insertion at widely separated points in a classical fluid with short range interactions factorizes.

(b) Explicit results for the condensate fraction of strongly correlated Bose fluids may be obtained from a generalization of the Widom particle insertion method in classical fluids via Eq. (10). In particular, using the known value $\tilde{\mu}_{\rm ex}$ for the excess chemical potential of a gas of hard spheres, the prediction of a zero temperature condensate fraction $n_0/n \simeq 0.08$ in liquid ^{4}He by Penrose and Onsager is close to the value obtained via path integral Monte Carlo methods [6].

Both conclusions are correct, however their derivation based on the mapping (5) is moot. A simple reason for this becomes evident from the fact that path integral Monte Carlo calculations of the hard sphere Bose fluid by Grüter et al. [20] show that its ground state is a non-superfluid crystal beyond $\eta \simeq 0.12$. The hard sphere system is thus no longer a fluid at ^{4}He densities. A more fundamental problem with Feynman's Ansatz connecting the many-body wave function to the square root of the distribution function of particle positions in a classical fluid is revealed by considering the limit of a dilute gas. The thermodynamic properties of the classical reference fluid may then be obtained from a virial expansion. For the specific case of a hard sphere system, using (11) to leading order in $\eta \ll 1$, this results in $n_0/n = 1 - 8\eta + \ldots$. The deviation of the condensate fraction from the ideal Bose gas limit is thus found to be linear in the density n. This contradicts the classic Bogoliubov result [21]

$$n_0 = n - \frac{8n}{3}\left(na^3/\pi\right)^{1/2} + \ldots = n - \frac{\sqrt{2}}{12\pi^2\,\xi^3} + \ldots \tag{12}$$

where the correction to n_0/n due to interactions scales with the square root of the density, a prediction that was verified experimentally by Lopes et al. [22]. In the limit of a dilute gas, the healing length $\xi = (8\pi na)^{-1/2}$ only depends on the density and the scattering length $a > 0$ as a single parameter characterizing the interaction. The physical origin of the discrepancy between Bogoliubov theory and a naive virial expansion is hidden in the fact that the interactions in the classical reference fluid underlying the representation (5) can not be of short range. Indeed, for any classical compressible fluid, the static structure factor $S_{\rm cl}(q \to 0) = (\partial n/\partial\tilde{\mu})/n$ is finite in the limit of vanishing wave vector. By contrast, a compressible quantum fluid at zero temperature has a static structure factor

$$S(q \to 0) = |\boldsymbol{q}|\xi/\sqrt{2} + \ldots \quad \to \quad g^{(2)}(r \to \infty) = 1 - \begin{cases} \frac{\xi}{\pi^2\sqrt{2}\,nr^4} & \text{in } d=3 \\ \frac{\xi}{2\pi\sqrt{2}\,n_2r^3} & \text{in } d=2 \end{cases} \tag{13}$$

which vanishes in a non-analytic manner. For a fluid of bosons, the associated characteristic length ξ is fixed by the sound velocity c_s via $\xi = \hbar/(\sqrt{2}mc_s)$, a relation which in fact holds for arbitrary strength of the interactions. This is a consequence of the fact that the Feynman-Bijl single mode result $E_q = \varepsilon_q/S(q) \to \hbar c_s q$ for the excitation energy becomes exact in the limit of small wave vectors [23], as will be discussed in more detail in Lecture III. Due to $S_{\rm cl}(q=0) \neq 0$, the Ansatz (5) does not describe correctly the long wavelength physics and therefore fails to reproduce the Bogoliubov result in the dilute limit. The

[3] See chapter 3.9 in the book by Hansen and McDonald [19]. Note that the expression (11) is applicable only in the fluid phase for $\eta < 0.49$ beyond which the equilibrium state of the classical hard sphere system is a crystal with an fcc-lattice structure, reaching close packing at $\eta_{\rm cp} = \pi\sqrt{2}/6 \simeq 0.74$.

excellent agreement of the prediction $n_0/n \simeq 0.08$ for the condensate fraction of ^{4}He with precise ab initio results which is obtained by using this mapping must therefore be considered as fortuitous. Formally, the behavior $S(q \to 0) \to |\boldsymbol{q}|\,\xi/\sqrt{2}$ can be enforced in a finite temperature classical fluid by adding long range repulsive two-body interactions $\tilde{v}(r) \to 1/(\sqrt{2}\pi^2 n\xi\, r^2)$ (or $\tilde{v}(r) \to 1/(\sqrt{2}\pi n_2\xi\, r)$ in two dimensions), as pointed out by Reatto and Chester [24]. Apart from the required knowledge of the effective healing length ξ or the associated sound velocity c_s, however, this interaction is not only density dependent but decays to zero so slowly that the condition (3) for the existence of a proper thermodynamic limit is violated. Whether the expression (10) for the condensate fraction in strongly interacting Bose fluids can be extended to cover a situation where the short distance scale σ can be taken to zero while the two-body scattering length a is of the order of the mean interparticle spacing or even infinite, is an open problem. It is of current interest in view of recent measurements of dimensionless ratios which characterize the unitary Bose gas, whose condensate fraction is estimated to be $n_0/n \simeq 0.2$ [25].

Bogoliubov theory as an internal Josephson effect in momentum space At this point, following in part Lectures by Nozières [26], it is instructive to add a few remarks regarding the Bogoliubov approach which are not discussed in standard textbooks. The approach relies on replacing the annihilation operator $\hat{b}_0 \to z$ for vanishing momentum by a complex number z (or $\bar{z}$ for $\hat{b}_0^\dagger$) and neglecting contributions to the interaction part of the second quantized form of the Hamiltonian (2) which contain only operators with finite momentum (for a discussion of why a replacement of operators by a c-number still gives the correct thermodynamics see Lieb et al. [27]) As a result, the Hamiltonian is reduced to a quadratic one

$$\hat{H}_{\rm Bog} = E_H + \sum_{q\neq 0}\left(\varepsilon_q + n_0 V(q)\right)\hat{b}_q^\dagger \hat{b}_q + \frac{1}{2V}\sum_{q\neq 0} V(q)\,\left(\bar{z}^2\,\hat{b}_q\hat{b}_{-q} + z^2\,\hat{b}_{-q}^\dagger\hat{b}_q^\dagger\right) \;\;\to\;\; E_{\rm Bog} + \sum_{q\neq 0} E_q \hat{\alpha}_q^\dagger\hat{\alpha}_q \tag{14}$$

which may be diagonalized by introducing a set of bosonic quasiparticles. Here, $n_0 = |z|^2/V$ is the condensate density and $V(q)$ is the Fourier transform of the two-particle interaction, which is assumed to be positive. Its value $g^{(0)} = V(q=0) > 0$ at vanishing momentum determines the Hartree energy $E_H = N\cdot g^{(0)}n/2$. The Hamiltonian (14) is a bosonic version of the reduced BCS-Hamiltonian for fermions. Provided that the phase of the complex number z can be choosen to vanish, the associated gap function $\Delta_q \equiv n_0\, V(q)$ is real and positive. As will be shown below, this is always possible, however a choice for the phase of z also fixes the phase associated with pairs $(q,-q)$ of particles in the depletion. Following the notation of standard textbooks [28], the operators $\hat{\alpha}_q^\dagger$ which create the bosonic quasiparticles with momentum q are connected with the corresponding operators $\hat{b}_q^\dagger$ of the underlying bosons by

$$\hat{\alpha}_q^\dagger = u_q\hat{b}_q^\dagger + v_q\hat{b}_{-q} \quad\leftrightarrow\quad \hat{b}_q^\dagger = u_q\hat{\alpha}_q^\dagger - v_q\hat{\alpha}_{-q} \quad \text{with}\;\; u_q^2 - v_q^2 = 1\,. \tag{15}$$

The amplitudes $u_q = \cosh\theta_q$ and $v_q = \sinh\theta_q$ are conveniently parametrized by a real rotation angle θ_q, depending only on the magnitude $q=|\boldsymbol{q}|$ of the wavevector. The ground state of the Hamiltonian (14) is defined by the condition $\hat{\alpha}_q|\Psi_{\rm Bog}\rangle \equiv 0$ of being the vacuum state for quasiparticles at all $q \neq 0$. It may be written in the form

$$|\Psi_{\rm Bog}\rangle = |z,\{\lambda_q\}\rangle = e^{-|z|^2/2}\prod_{q\neq 0}(1-|\lambda_q|^2)^{1/2}\exp\left(z\,\hat{b}_0^\dagger + \sum_{q\neq 0}\lambda_q\hat{b}_q^\dagger\hat{b}_{-q}^\dagger\right)|0\rangle \tag{16}$$

of a product of a coherent state for the condensate with one involving pairs $(q,-q)$ with vanishing total momentum for the depletion. Indeed, choosing $\lambda_q = -v_q/u_q$, the state (16) has a vanishing number of quasiparticles because

$$\left[\hat{b}_q, \exp\left(\lambda_q\hat{b}_q^\dagger\hat{b}_{-q}^\dagger\right)\right] = \lambda_q\hat{b}_{-q}^\dagger\,\exp\left(\lambda_q\hat{b}_q^\dagger\hat{b}_{-q}^\dagger\right) \;\;\to\;\; \hat{\alpha}_q\,|\Psi_{\rm Bog}\rangle = (u_q\lambda_q + v_q)\,\hat{b}_{-q}^\dagger|\Psi_{\rm Bog}\rangle \equiv 0 \;\;\text{if}\;\; u_q\lambda_q + v_q = 0\,.$$

Note that the condition $\hat{\alpha}_q|\Psi_{\rm Bog}\rangle \equiv 0$ only involves finite momenta $q \neq 0$. The precise form choosen for the condensate wavefunction is thus left open. Taking this to be a simple coherent state $|z\rangle$ is just a convenient choice. For a given total number N of bosons, which is fixed only on average [4], the associated parameter $z \to z_\lambda$ is eliminated as an independent variable through the constraint $N_0 = |z|^2 = N - \sum_{q\neq 0}\langle\hat{b}_q^\dagger\hat{b}_q\rangle$. The problem is thus reduced to determining the variables λ_q. Now, in order to understand the underlying physics and the generality of Bogoliubov's approach, it is instructive to determine the expectation value of the Bogoliubov Hamiltonian in the normalized state (16), using a parametrization of the - in general complex - variables $\lambda_q = \tanh\theta_q\exp i\varphi_q$ in terms of a real parameter θ_q and a phase φ_q, The necessary expectation values are $\langle\hat{b}_q^\dagger\hat{b}_q\rangle = \sinh^2\theta_q$ for the average occupation number of bosons with finite momentum and a nonzero 'anomalous' expectation

[4] For a strictly number conserving formulation of Bogoliubov theory see Girardeau [29] and the review by Leggett [30]. Note also that in a quantum optics context, Eq. (16) describes a two-mode squeezed state, see e.g. Walls and Milburn [31]. This analogy is discussed by Haque and Ruckenstein [32].

value $\langle \hat{b}_q \hat{b}_{-q} \rangle = \sinh\theta_q \cosh\theta_q \exp i\varphi_q$ which depends on the phase of λ_q. Defining a phase φ_c for pairs of particles in the condensate by $z^2 = N_0 \exp i\varphi_c$, the expectation value of the Bogoliubov Hamiltonian in the state (16) has the form

$$\langle z_\lambda, \{\lambda_q\} | \hat{H}_{\rm Bog} | z_\lambda, \{\lambda_q\} \rangle = E_H + \sum_{q\neq 0} \left[\xi_q \sinh^2\theta_q + \Delta_q \sinh\theta_q \cosh\theta_q \cdot \cos(\varphi_c - \varphi_q) \right] . \tag{17}$$

Here, $\xi_q = \varepsilon_q + \Delta_q$ is the single-particle energy within a Hartree-Fock approximation. It approaches a constant $\Delta_0 = n_0 g^{(0)}$ as $q \to 0$ and thus would lead to a finite excitation gap. This is at variance with the expected gapless nature of the excitations associated with the breaking of the global continuous symmetry $\hat{b}_q \to \hat{b}_q \exp(i\varphi)$ which is still present in (14). To see how the actual gapless excitations $E_q \to \hbar c_s q + \ldots$ arise within the Bogoliubov approach, it is necessary to include the phase dependent contribution to the energy (17). Apparently, this term is minimized by choosing a fixed relative and momentum independent phase $\Delta\varphi = \varphi_c - \varphi_q = \pi$. In an interacting BEC, therefore, there is an effective internal π-Josephson junction in momentum space between pairs of particles in the condensate and those with opposite momentum in the depletion (note that pairs are necessary because the ground state must have zero momentum). The associated phase dependent coupling energy $E_J \cos\Delta\varphi = -E_J = -\sum_{q\neq 0} \Delta_q \sinh\theta_q \cosh\theta_q$ is negative despite the fact that the underlying interaction is purely repulsive. This is analogous to what happens in the effective π-Josephson junction at the interface between a d-wave and an s-wave superconductor, where tunneling occurs between gaps which are positive on the s-wave and negative on the d-wave side, a setup, which has been used to determine the non-trivial nature of pairing in high-temperature superconductors by Wollman et al. [33]. On a formal level, the relative phase $\Delta\varphi = \pi$ between the condensate and the depletion just accounts for the minus sign which appears in $\lambda_q = -v_q/u_q$. The underlying physics, however, has a number of important and not widely appreciated consequences:

(a) The internal Josephson coupling between pairs of particles in the condensate and those in the depletion with opposite momentum is both necessary and sufficient for the generic behavior (13) of the static structure factor of a compressible Bose fluid and thus eventually for the gapless nature of the excitation spectrum. It explains, moreover, the fact that the ground state is fully superfluid despite a condensate density which might be below ten percent as in ^{4}He.

To see this, consider the static structure factor $S(q) = \langle \hat{\rho}_q^\dagger \hat{\rho}_q \rangle$ which involves the normalized density fluctuation operator $\hat{\rho}_q^\dagger = \sum_k \hat{b}_{k+q}^\dagger \hat{b}_k / \sqrt{N}$. Within the Bogoliubov approach, this can be calculated exactly to zeroth order in the small parameter $(na^3)^{1/2}$ by restricting $\hat{\rho}_q^\dagger \simeq \hat{b}_q^\dagger + \hat{b}_{-q}$ to those contributions which involve $\hat{b}_0 \to z$, which is conveniently choosen to be real. As a result, one obtains

$$S_{\rm Bog}(q) = \langle \Psi_{\rm Bog} | \hat{\rho}_q^\dagger \hat{\rho}_q | \Psi_{\rm Bog} \rangle = \left. \frac{1 + 2\tanh\theta_q \cos\varphi_q + \tanh^2\theta_q}{1 - \tanh^2\theta_q} \right|_{\varphi_q = \pi} = \exp -2\theta_q \xrightarrow[\Delta_q = \rm const]{} (1 + 2/q^2\xi^2)^{-1/2} . \tag{18}$$

Here, in the final form of the expression, we have used that fixing $\Delta\varphi = \pi$ at its optimum value, a minimization of the energy (17) with respect to the remaining variables θ_q leads to $\tanh 2\theta_q = \Delta_q/\xi_q$. This determines the momentum dependence of the static structure factor $S_{\rm Bog}(q) = \exp -2\theta_q = (1 + 2\Delta_q/\varepsilon_q)^{-1/2}$. In particular, defining the sound velocity via $\Delta_0 = n_0 g^{(0)} \to mc_s^2$ and an associated characteristic length ξ via $\xi = \hbar/(\sqrt{2}\, mc_s)$, its behavior at small momentum is identical with the one given in Eq. (13). The Bogoliubov approach thus provides a proper description of the pair distribution function at long distances of any compressible Bose fluid. Evidently, it is precisely the minus sign $\cos\varphi_q = -1$ associated with the internal Josephson effect which guarantees that the leading contributions at small q in the numerator of the static structure factor (18) precisely cancel.

In order to understand why a BEC is fully superfluid at zero temperature despite the fact that the fraction f_0 of particles in the condensate may be much less than one, one needs to show that the superfluid fraction $f_s = N_s/N$ is equal to one at $T = 0$. Here, as discussed further below, N_s is defined in such a way that $N_s \cdot \hbar^2 \boldsymbol{Q}^2/2m$ is the increase in the total energy of a state in which the Bose fluid is set into motion with a finite momentum $\boldsymbol{Q}$. Now, as a result of the Josephson coupling between the condensate and the depletion through an extensive energy E_J, this momentum is carried not only by the particles in the condensate but the *complete* momentum distribution is translated by $\boldsymbol{Q}$, giving rise to a mass current density $n_s \cdot \hbar\boldsymbol{Q}$ with $n_s = n$. The particles in the depletion are rigidly dragged along, with pairs now at $\boldsymbol{q} + \boldsymbol{Q}, -\boldsymbol{q} + \boldsymbol{Q}$. As a result, the system is a perfect superfluid at zero temperature irrespective of the value of the condensate fraction f_0 as long as this is finite.

(b) The well defined relative phase between the condensate and the depletion is the origin of anomalously large fluctuations in the respective particle numbers $\hat{N}_0$ or $\hat{N}' = \Sigma_{q\neq 0} \hat{n}_q$ which are enhanced by a factor L/ξ or L/λ_T at finite temperature compared to the situation in the absence of the coherent coupling (here L is the system size and $\lambda_T = \hbar\sqrt{2\pi/mk_B T}$ the thermal wavelength).

Focussing on the zero temperature limit, the fluctuations of the number of particles in the condensate within Bogoliubov

$$\mathrm{Var}\, \hat{N}_0 = \mathrm{Var}\, \hat{N}' = 2\,\Sigma_{q\neq 0} \langle \hat{n}_q \rangle (1 + \langle \hat{n}_q \rangle) = 2\,\Sigma_{q\neq 0} u_q^2 v_q^2 = V/(8\pi\sqrt{2}\,\xi^3) , \tag{19}$$

have been determined by Giorgini et al. [34]. Here, the prefactor two is a direct consequence of pairing in states $(q, -q)$. At first sight, the linear scaling with the volume is the expected behavior for the fluctuations of an extensive variable in thermodynamics. This argument is misleading, however, because at zero temperature the fluctuations of the number of particles enclosed in a volume V are basically a surface effect, obeying an area law $\mathrm{Var}\,\hat{N} \simeq (V/\xi^3)^{2/3} \ln (V/\xi^3)$ which is modified by a logarithmic factor [34]. A behavior of this type is generic for a compressible system in contact with a reservoir, where exchange of particles occurs in an incoherent fashion. By contrast, in the presence of a coherent coupling between system and reservoir, the number fluctuations are enhanced by a factor $\sim V^{1/3}$ and are thus of an extensive nature even at zero temperature. The large enhancement of number fluctuations is therefore a consequence of the internal Josephson effect connecting the condensate and the depletion. It also shows up at finite temperature, where $\mathrm{Var}\,\hat{N}_0(T) \simeq (L/\lambda_T)^4 \sim V^{4/3}$ is again a factor $L/\lambda_T \sim V^{1/3}$ larger than what is expected for a standard extensive variable in thermodynamics [34]. This result is in fact not confined to a Bogoliubov approximation but is a generic feature of BEC's with an arbitrary strength of the interaction [35]. More generally, anomalously large fluctuations of the order parameter appear for all phases with a broken continuous symmetry [36].

(c) With a proper renormalization of the parameters, in particular the replacement $\Delta_0 = n_0 g^{(0)} \to mc_s^2$ of the bare gap parameter by the square of the exact velocity of sound, Bogoliubov theory provides an asympotically exact description of the low-energy physics of Bose fluids with an arbitrary strength of the interactions.

To appreciate this point, it should be noted first that even for dilute BEC's the parameter $g^{(0)} = V(q=0) = 4\pi\hbar^2\, a^{(0)}/m$ contains the scattering length associated with the two-body interaction $V(\boldsymbol{x})$ only at the Born approximation level $a^{(0)}$. It is standard practice to replace this by the exact value a, using e.g. a pseudopotential Hamiltonian as introduced by Huang and Yang [37]. A more general approach which starts with a bare microscopic action and allows to properly account for the low energy constants associated with the two- and three-body and in principle even higher order interactions is provided by the method of effective potentials, as will be used in the context of the gas-liquid transition in Eq. (29) below. In this more modern formulation, the well known LHY-correction $E_{\mathrm{Bog}} = N \cdot gn/2\,(1 + 128\sqrt{na^3}/15\pi + \ldots)$ [38] to the mean-field ground state energy appears as the properly regularized one-loop contribution $(1/2)\sum_q E_q$ to the Coleman-Weinberg potential which arises from the zero point energy of the Bogoliubov excitations. Concerning the replacement $\Delta_0 \to mc_s^2$ within the Bogoliubov formalism, it is straightforward to see that it accounts properly for the correct linear behavior $E_q = (\xi_q^2 - \Delta_q^2)^{1/2} \to \hbar c_s q$ of the excitation spectrum at low energy as well as the singular nature of the ground state momentum distribution $n_q = \sinh^2\theta_q \to (mc_s/2\hbar q)$ which - up to a renormalization factor n_0/n - has been shown to be an exact result by Gavoret and Nozières [39].

As a final point in this context, we mention a fundamental issue associated with many-body wave functions in general. In fact, their detailed form becomes meaningless in practice for particle numbers beyond $N \simeq 10^3$, a problem which has been called the van Vleck catastrophy by Kohn [40]. To understand the origin of this problem, it is instructive to consider the overlap between two many-body wave functions for different interaction strengths specified e.g. by adjacent values a and a' of the scattering length. Quite generally, the magnitude of this overlap appears only at second order in the deviation $\delta a = a' - a$ but decreases exponentially with the number of particles. The sensitivity of a many-body wavefunction to a small change δa in some parameter may thus be characterized by an intensive fidelity susceptibility χ_F which is defined by $|\langle\Psi(a)|\Psi(a')\rangle| = \exp\left(-\frac{1}{2}N\chi_F(\delta a)^2\right)$. By dimensional analysis, the fidelity susceptibility $\chi_F = 1/\ell_F^2$ defines a characteristic scale ℓ_F for the parameter a. As a result, knowledge of the many-body wave function with an accuracy close to one requires to know the microscopic parameter a with an accuracy $|\delta a| \ll \ell_F/\sqrt{N}$ which is obviously impossible for large particle numbers. This is one way of expressing the exponential wall encountered in determining many-body wave functions, emphasized by Kohn [40]. The exactly known Bogoliubov wave function (16) serves as a concrete illustration of these ideas. Up to second order in δa, the overlap of two such states is given by

$$|\langle z_\lambda, \{\lambda_q\}|z'_\lambda, \{\lambda'_q\}\rangle| = \exp\left(-\frac{1}{2}N\chi_F(\delta a)^2\right) \;\text{ with }\; \chi_F = \left(\partial_a\sqrt{f_0}\right)^2 + \frac{1}{n}\int_q \frac{(\partial_a\lambda_q)^2}{(1-|\lambda_q|^2)^2}\,. \tag{20}$$

The first term in the fidelity susceptibility arises from the overlap of the coherent states for the condensate. Using the leading order Bogoliubov result (12) for the depletion, it is given by $\chi_F^{(0)} = 1/(2\pi^2\xi^2) \simeq na$. The characteristic scale which determines how an uncertainty in the scattering length affects the accuracy of the many-body state thus appears to be the healing length ξ. Surprisingly, this conclusion is changed fundamentally by including the second contribution to χ_F in Eq. (20) which arises from the overlap of the product of two-mode squeezed states. Using that both in the regime $q\xi \ll 1$, where $\lambda_q \to -1 + \sqrt{2}q\xi$ and for $q\xi \gg 1$, where $\lambda_q \to -1/(2q^2\xi^2)$, the derivative $\partial_a\lambda_q$ with respect to the scattering length can be easily determined, it turns out that in the relevant limit $(na^3)^{1/2} \ll 1$, the fidelity susceptibility $\chi_F \simeq (n/a)^{1/2}$ is dominated by the second contribution, which diverges for vanishing scattering length. This divergence is a signature of a quantum phase transition from a gaseous to a liquid ground state of Bose fluids at $a = 0$ which will be discussed in more detail below. More generally, as shown by Wang et al. [41], the fidelity susceptibility can be calculated efficiently via Quantum Monte Carlo methods in cases where no explicit results for the many-body wave function are available. In particular, it serves as an indicator of putative quantum phase transitions without an a priori knowledge of the order involved.

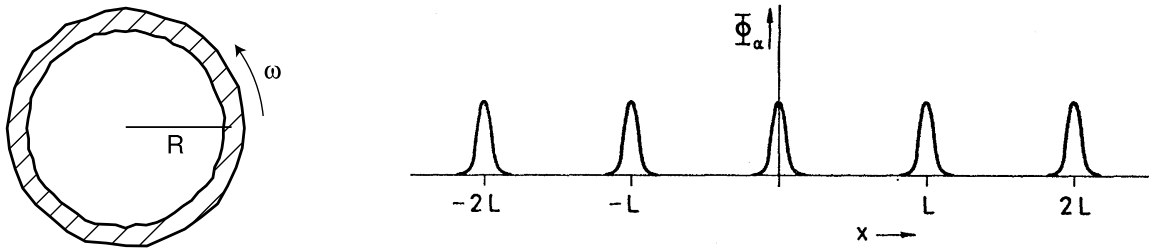

FIG. 4: A rotating Bose fluid in a ring geometry with non-perfect walls. The right Figure shows a schematic localized many-body wave function $\Phi_\alpha(x)$ on a ring with circumference L as a function of one of the coordinates. The Figure is taken from Ref. [42].

Topological nature of many-body wave functions of superfluids As a consequence of the van Vleck catastrophy, the criterion for superfluidity cannot depend on the precise form of the many-body wave function but only on some long distance or topological properties. This point was first elucidated by Kohn [42] in the context of a quite general characterization of insulating ground states of interacting Fermi systems. Kohn's basic idea was to consider the many-body problem in a ring geometry and in the presence of a finite magnetic flux. For electrons with charge $-e$, this is a standard Aharanov-Bohm type setup which had been analyzed earlier by Byers and Yang [43] in their quite general proof of flux quantization in superconducting rings. In the case of neutral particles, an effective flux arises in a situation where the many-body system is enclosed between two concentric cylinders with nearly equal radii R, co-rotating with an angular frequency $\boldsymbol{\omega} = \omega \boldsymbol{e}_z$. As indicated in Fig. 4, the walls are assumed to violate perfect cylindrical symmetry to allow for the transfer of angular momentum to the fluid. In the rotating frame, the problem is stationary, however the non-inertial frame gives rise to an effective gauge potential $\boldsymbol{A}(\boldsymbol{x}) = m\,\boldsymbol{\omega} \wedge \boldsymbol{x}$ which appears in the kinetic energy part $\sum_j(\hat{\boldsymbol{p}}_j - \boldsymbol{A}(\boldsymbol{x}_j))^2/2m$ of the Hamiltonian. Formally, the gauge potential can be eliminated by a gauge transformation $\psi^{(\theta)}(\boldsymbol{x}_1 \dots \boldsymbol{x}_N) = \exp\left[-i(mR\omega/\hbar)\sum_j x_j\right]\psi(\boldsymbol{x}_1 \dots \boldsymbol{x}_N)$ to a new many-body wave function $\psi^{(\theta)}$ which obeys the Schrödinger equation in the absence of $\boldsymbol{A}$. This function, however, is no longer single-valued. Instead, it changes by a phase factor if any one of the particles is taken around the ring according to

$$\psi^{(\theta)}(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots \boldsymbol{x}_i + L, \dots \boldsymbol{x}_N) = e^{-i\theta}\,\psi^{(\theta)}(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots \boldsymbol{x}_i, \dots \boldsymbol{x}_N) \quad \forall \quad i = 1 \dots N \tag{21}$$

with $\theta = 2\pi m R^2\omega/\hbar$. Here, $L = 2\pi R$ is the circumference and $\boldsymbol{x}_i + L$ means that the i-th coordinate is taken around the ring once, with transverse coordinates and possible other degrees of freedom like spin in the case of fermions held fixed. The twist (21) in the boundary condition leads to a spectrum of eigenvalues $E_\alpha(\theta)$ which will in general depend on θ, giving rise to a phase dependent equilibrium free energy $F(\theta)$ in the stationary, rotating system. As realized by Byers and Yang [43], $F(\theta)$ is an even and periodic function $F(\theta + 2\pi) = F(\theta)$, irrespective of the strength of the interactions provided these are time reversal invariant. It can therefore be expanded in a Fourier series

$$\Delta F(\theta) = F(\theta) - F(\theta = 0) = \sum_{l=1}^{\infty} F_l\,[1 - \cos(l\,\theta)] \;\rightarrow\; L_z^{\rm rot}(\theta) = -\frac{\partial F(\theta)}{\partial \omega} \xrightarrow[\omega \to 0]{} -\left(\frac{LmR}{\hbar}\right)^2 \sum_{l=1}^{\infty} l^2 F_l \cdot \omega = -(n_s/n)\,L_z^{(0)} \tag{22}$$

whose derivative with respect to ω determines the kinematic angular momentum $L_z^{\rm rot}$ in the rotating frame. The superfluid fraction n_s/n in this setup is now defined by expressing $L_z^{\rm rot} = -(n_s/n)\,L_z^{(0)}$ in terms of the characteristic angular momentum $L_z^{(0)} = I_{\rm cl}\,\omega$ in a situation where a fluid is fully carried along by the walls at angular frequency ω, with $I_{cl} = NmR^2$ the associated moment of inertia. Physically, a finite and negative angular momentum $L_z^{\rm rot} = -(n_s/n)\,L_z^{(0)}$ in the rotating frame implies that a fraction n_s/n of the superfluid stays at rest in the lab frame for small angular frequencies $\omega \ll \hbar/mR^2$. As a result, the apparent moment of inertia is smaller than that of classical rigid body rotation. The property of a non-classical rotational inertia (NCRI) has been introduced as a definition of superfluidity in a paper by Leggett [44] where he discussed the possibility of a finite n_s even in a solid, an issue that will be investigated in more detail below. In the context of cold gases, the prediction that a superfluid does not rotate with its walls for small rotation frequencies has been demonstrated in experiments at the ENS [45, 46]: a trapped BEC in the presence of a small, non-symmetric perturbation remains at zero angular momentum below a finite critical rotation frequency. A direct signature for the existence of NCRI is provided by the so-called scissors mode in BEC's with anisotropic confinement $\omega_x \neq \omega_y$ in the plane perpendicular to the rotation. For superfluid flow, their effective moment of inertia $I_{\rm SF} = \delta^2\,I_{\rm cl}$ is smaller than the classical rigid body value $I_{\rm cl} = Nm\,\langle X^2 + Y^2\rangle$ by a factor $\delta^2 < 1$ which depends on the deformation parameter $\delta = \langle X^2 - Y^2\rangle/\langle X^2 + Y^2\rangle$. As predicted by Guéry-Odelin and Stringari [47], the fact that angular momentum in an anisotropic trap is not conserved gives rise to an oscillation of the gas after a sudden rotation of the trap around the new equilibrium position with frequency $\omega_{\rm scis} = (\omega_x^2 + \omega_y^2)^{1/2}$ which is absent in the normal phase, in perfect agreement with experiment [48].

Eq. (22) shows that a finite superfluid fraction requires the existence of a rigidity parameter γ with dimension energy per length such that the second moment $\sum_l l^2 F_l \simeq \gamma \cdot A_\perp/L$ of the Fourier amplitudes scales linearly with the cross section area $A_\perp$ and has a slow power law decay $\sim 1/L$ with the circumference of the ring. In the limit $A_\perp, L \to \infty$, this gives rise to a superfluid density $n_s = \gamma m/\hbar^2$ which is independent of the sample dimension. To define the underlying rigidity in a more general form and, moreover, to describe states of a superfluid with finite currents, it is useful to introduce a slowly varying local phase $\varphi(\boldsymbol{x})$ on scales much larger than the interparticle spacing which is connected with the total phase difference between two arbitrary points by $\theta = \int ds\, \nabla\varphi(\boldsymbol{x})$. The free energy increase due to a finite value of $\nabla\varphi(\boldsymbol{x})$ can then be expressed in a local form

$$\Delta F[\varphi(\boldsymbol{x})] = \frac{\gamma}{2}\int_x (\nabla\varphi(\boldsymbol{x}))^2 \quad \text{with} \quad \gamma = \frac{\hbar^2 n_s}{m} = \frac{L^2}{V}\frac{\partial^2 \Delta F(\theta)}{\partial\theta^2}\bigg|_{\theta=0} \underset{\text{ring}}{\longrightarrow} \frac{L}{A_\perp}\sum_{l=1}^{\infty} l^2 F_l \tag{23}$$

which, however, hides the periodic dependence on θ stated in Eq. (22) [5]. Physically, a non-vanishing phase gradient corresponds to a finite superfluid velocity $\boldsymbol{v}_s = (\hbar/m)\nabla\varphi(\boldsymbol{x})$. The rigidity energy is thus just the kinetic energy of superfluid flow which may be present even in an equilibrium configuration (see the Appendix for a more detailed discussion). In the particular case of a uniform twist $\nabla\varphi(\boldsymbol{x}) = \boldsymbol{Q}$, Eq. (23) shows that the total number N_s of particles in the superfluid is defined by the increase $N_s \cdot \hbar^2\boldsymbol{Q}^2/2m$ in free energy if the whole fluid acquires a finite momentum $\boldsymbol{Q}$, as was used above in the context of the internal Josephson effect in the Bogoliubov approach. It is important to note that the definition (23) for superfluidity is based only on equilibrium properties and it also applies to finite systems. Obviously, however, it is quite different from the definition of BEC via the concept of ODLRO, as stated in Eq. (1). Yet, it turns out, that the two phenomena are intimately connected. In fact, superfluidity in the sense defined in Eq. (23) is the more general phenomenon. On a qualitative level, the connection between a finite value of the superfluid stiffness γ and the presence of ODLRO may be understood by using the representation $\hat\psi(\boldsymbol{x}) \simeq \sqrt{\tilde n_0}\exp i\hat\varphi(\boldsymbol{x})$ of the Bose field operator in terms of a finite bare condensate density $\tilde n_0$ and the phase operator $\hat\varphi(\boldsymbol{x})$. The asymptotic decay of $\rho^{(1)}(\boldsymbol{x},\boldsymbol{x}') = \tilde n_0 \exp[-\delta\varphi^2(\boldsymbol{x},\boldsymbol{x}')/2]$ is then determined by the mean square fluctuations $\delta\varphi^2(\boldsymbol{x},\boldsymbol{x}') = \langle(\hat\varphi(\boldsymbol{x}) - \hat\varphi(\boldsymbol{x}'))^2\rangle$ of the phase difference between points separated by $|\boldsymbol{x}-\boldsymbol{x}'|$. Using the effective Hamiltonian (23) together with the assumption of a finite compressibility it is possible to show (see e.g. the Appendix in Ref. [49]) that the phase fluctuations remain finite in the limit of infinite separation in three dimensions. As a result, $\gamma \neq 0$ implies ODLRO with a condensate density $n_0 = \tilde n_0 \exp[-\delta\varphi^2(\infty)/2]$. In two dimensions, this result only holds at $T = 0$, while $\delta\varphi^2(\boldsymbol{x},\boldsymbol{x}') \to 2\eta \ln|\boldsymbol{x}-\boldsymbol{x}'|$ diverges logarithmically at finite temperatures below the BKT-transition, where $\eta(T_{\mathrm{BKT}}) = 1/4$. This leads to an algebraic decay $\rho^{(1)}(\boldsymbol{x},\boldsymbol{x}') \sim |\boldsymbol{x}-\boldsymbol{x}'|^{-\eta}$, consistent with the Mermin-Wagner-Hohenberg theorem, which states that no long range order is possible in two dimensions if $T \neq 0$ in the case of a continuous symmetry. A similar behavior, due to quantum rather than thermal phase fluctuations, applies in one dimension at zero temperature.

In the following, it will be shown that the definition of superfluidity based on Eqs. (22) and (23) allows to characterize superfluids in terms of a topological property of the many-body wave function which implies, in particular, that ground states of bosons are always superfluid provided they have a uniform density. The argument relies on the geometry introduced above, where the many-particle configuration space is an N-torus $\mathbb{T}^N = S^1 \otimes \cdots \otimes S^1$ with respect to motion around the ring. The dependence of the energy levels $E_\alpha(\theta)$ and the associated free energy $F(\theta)$ of the many-body system in the rotating frame or of the charged system in the presence of a finite magnetic flux is determined by the change in energy induced by the twist in Eq. (21) associated with closed paths in configuration space. To single out the dependence on the variable θ, it is useful to consider the representation of the partition function of the many-body system in terms of a Feynman propagator over closed paths $\{\boldsymbol{x}_j\} \to \{\boldsymbol{x}_j\}$ in imaginary time $\beta\hbar$. Since the configuration space is multiply connected, this propagator is a sum over the different elements of the first homotopy group $\pi_1(\mathbb{T}^N) = \mathbb{Z}^N$ of the N-torus which are labelled by the set of N integer winding numbers $m_j \in \mathbb{Z}$. Physically they correspond to taking any of the $j = 1 \ldots N$ particles around the ring m_j times. As shown by Pollock and Ceperley [50], the change in free energy due to the twist in the boundary condition is determined by the characteristic function

$$\exp(-\beta\Delta F(\theta)) = \sum_{\{m_j \in \mathbb{Z}\}} e^{-iM\theta}\, p(m_1 \ldots m_N; \beta) \quad \text{with} \quad M = \sum_j m_j \tag{24}$$

of the winding number probability distribution $p(m_1 \ldots m_N; \beta)$ in the *absence* of the twist. Considering in particular the limit where the temperature approaches zero, the question of whether the ground state energy in the rotating frame exhibits a non-trivial dependence on the twist θ is determined by the connectedness properties of the ground state wave function. In the ground breaking papers on this subject by Kohn [42] and Leggett [44, 51], two limiting cases were considered:

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[5] The periodicity in the variable θ is important e.g. for understanding the exactness of flux-quantization in superconducting rings with a thickness much larger than the London penetration depth. Using $\theta = 2\pi\phi/\phi_0$, this relies on the fact that the large energy associated with the Fourier coefficient $F_{l=2} \simeq \gamma A_\perp/L$ forces $\cos(2 \cdot 2\pi\phi/\phi_0) = 1$ with negligible fluctuations. The magnetic flux ϕ is thus pinned at an integer number times the flux quantum $\phi_0/2$ in superconductivity.

a) The wave function of the ground state is disconnected in the sense that on all closed paths with $M \neq 0$, there is at least one region where the wave function is exponentially small. In the presence of rotation, the modified boundary condition (21) can then be accomodated by adding the phase shift in precisely these regions. The resulting change in energy $\sim \exp(-L/\xi_{\rm loc})$ vanishes exponentially and thus the free energy $F(\theta) \simeq F(0)$ in the rotating frame becomes independent of the twist as $L \gg \xi_{\rm loc}$. This is the characterisation given by Kohn for insulators. Specifically, Kohn discussed electrons in a regular lattice with a set $\boldsymbol{R}_\nu$ of sites whose number is commensurate with those of the electrons. As indicated schematically in Fig. 4, they were described by exponentially localized Wannier functions $w(\boldsymbol{x} - \boldsymbol{R})$ at the single-particle level, leading to a disconnected many-body state.

b) The wave function $\psi_0(\boldsymbol{x}_1 \ldots \boldsymbol{x}_N)$ of the ground state is connected in the sense that there exist closed paths with non-vanishing total winding number $M \neq 0$ on which the magnitude $|\psi_0|$ is everywhere bounded below by a finite constant independent of both N and L. In this case, the system is a superfluid with a reduced moment of inertia because the twist leads to an energy increase $\sim \int |\psi_0|^2_{\rm min}(\nabla\varphi)^2$ of order $A_\perp/L$. For fluid ground states, the existence of closed paths of this type may be viewed as a consequence of the positivity of the many-body ground state wave function. As pointed out by Leggett [51], ground states of bosons with a uniform density are therefore always superfluid [6]. For non-uniform ground states like in a crystal, the positivity requirement, however, is not sufficient to infer the existence of a finite superfluid density because the minimum magnitude $|\psi_0|_{\rm min} \sim \exp(-L/\xi_{\rm loc})$ could be exponentially small as in insulators. In this situation, there is only an upper bound on the superfluid fraction which will be discussed in detail in Lecture II.

An important point to note in this context is that the magnitude $|M|$ of the relevant total winding numbers are of order one or two and not of order N because the relevant Fourier components F_l in Eq. (22) are $l = 1$ or $l = 2$ for standard Bose superfluids or superfluids of Fermion pairs, respectively. In physical terms, this requires that there are paths in the configuration space where the many-body wave function stays finite upon taking one or maybe two particles around the ring while the coordinates of the remaining $N-1$ particles are held fixed. Obviously, this is the case in the presence of ODLRO as defined in Eq. (1), which thus turns out to be a sufficient criterion for superfluidity. It is not a necessary one, however, and indeed as stated above, superfluidity is the more general phenomenon rather than BEC and the equivalent existence of ODLRO.

In the case of charged systems, the dependence of the eigenvalues in the presence of a non-trivial boundary condition (21) leads to a characterization of insulators or superconductors in terms of the so-called Drude weight [42]

$$D_s = \pi \lim_{\omega\to 0} \omega\, {\rm Im}\, \sigma(\omega) = \pi\, \frac{n_s e^2}{m} = \frac{e^2}{\hbar^2}\frac{\pi L^2}{V}\frac{\partial^2 \Delta F(\theta)}{\partial\theta^2}\bigg|_{\theta=0} \xrightarrow[L\to\infty]{} \begin{cases} \sim \exp(-L/\xi_{\rm loc}) & \text{insulator} \\ D_s \neq 0 & \text{superconductor} \end{cases} \tag{25}$$

For superconductors, this implies a $1/\omega$-singularity of strength D_s/π in the imaginary part of the frequency dependent conductivity which is precisely the content of the phenomenological first London equation. In the case of insulators, such a contribution is absent and the odd function ${\rm Im}\,\sigma(\omega)$ therefore vanishes linearly at low frequencies. However, this is also true in metals with a finite amount of disorder. The relevant distinction between metals and insulators shows up in the behavior of ${\rm Re}\,\sigma(\omega)$ as $\omega \to 0$: for any non-perfect metal, the real part of the conductivity has a finite value while ${\rm Re}\,\sigma(\omega) \sim \omega^2 \ln^{d+1}(\bar{\omega}/\omega)$ vanishes essentially quadratically in insulators. A discussion of how the empirical description of the different ground states in terms of the complex conductivity $\sigma(\omega)$ is reflected at the level of the Drude weight has been given by Scalapino et al. [52]. According to Eq. (23), the Drude weight at $T = 0$ is obtained from the curvature of the many-body ground state. This requires to follow the ground state adiabatically as a function of the twist θ. Now, it turns out that the characteristic magnitude θ_c of the twist at which another many-body level crosses or drops below the ground state varies like $\theta_c \sim 1/L^{d-1}$. In dimension $d > 1$, therefore, the order of limits $\theta \to 0$ and $L \to \infty$ matters: taking the second derivative of $E_0(\theta)$ with respect to θ first, and then sending $L \to \infty$ gives a Drude weight D. It differs from the D_s defined above, which involves the curvature of the envelope of the $E_\alpha(\theta)$ curves of individual many–body states ψ_α. Both D and D_s approach zero for an insulator and they are both finite in a superconductor. In the case of a metal with no disorder, however, D is finite while $D_s = 0$ [52]. A different way to see that there is no topological characterization of metallic or normal fluid states is revealed by the fact that the second moment $\sum_l l^2 F_l|_{\rm normal} \simeq (\hbar^2/mL)\, n\xi_t^2$ of the Fourier amplitudes in Eq.(22) still scales with $1/L$. The linear increase with the transverse area $A_\perp$ in the superfluid phase, however, is replaced by the square of a characteristic length ξ_t which appears in the momentum dependence $\chi_t(q) = \rho[1 - (q\xi_t)^2 \ldots]$ of the transverse current response, associated with diamagnetism in the charged case. The periodic dependence of $L_z^{\rm rot}$ on θ is still present and it describes the persistent currents in a normal metal ring predicted by Büttiker, Imry and Landauer [53]. The observed magnitude of the associated Fourier coefficients F_l agrees well with a model of non-interacting electrons [54], however the role of interactions in this context has remained controversial.

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[6] This conclusion no longer holds in the presence of a magnetic field, as shown for example by the incompressible Quantum Hall state of a half filled Landau level in two dimensions described by the Laughlin wave function $\Psi_{\rm L}(z_1, \ldots z_N) = \prod_{i<j}(z_i - z_j)^2 \cdot \exp - \sum_i |z_i|^2/4$, which has a uniform density $n(z) = 1/4\pi$.

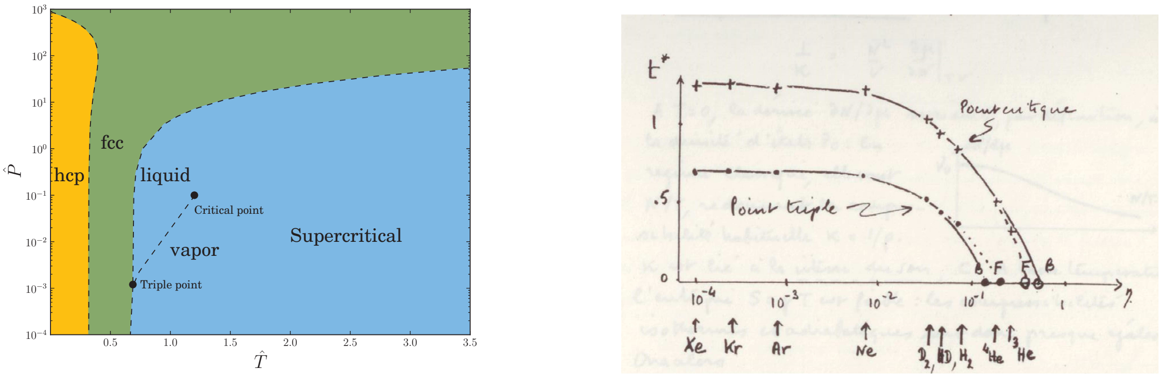


FIG. 5: The Figure on the left shows the phase diagram of particles with a Lennard-Jones interaction in the classical limit of a vanishing de Boer parameter $\Lambda_{\mathrm{dB}} = 0$ as determined by Travesset [55]. The dimensionless pressure and temperature are defined by $\hat{P} = p\sigma^3/\epsilon$ and $\hat{T} = k_B T/\epsilon$. The Figure on the right shows the dependence of the dimensionless temperature $\hat{T} \to t^*$ of the critical and the triple point as a function of the square $\eta = \Lambda_{\mathrm{dB}}^2$ of the de Boer parameter. It is taken from Lectures given by P. Nozières at the Collège de France in 1983.

Quantum-unbinding at a zero temperature liquid-gas transition Following recent work [56, 57], we will discuss the liquid-to-gas quantum unbinding transition in Bose fluids induced by an increasing strength of the zero point fluctuations. The existence of such a transition is indicated in Fig. 5, where the dimensionless temperature of both the triple and the critical point are shown as a function of the square of the de Boer parameter. The transition from a solid to a liquid ground state occurs when the triple point vanishes. It is first order and the associated critical de Boer parameter $\Lambda_{\mathrm{dB}}^{\mathrm{c,solid}} \simeq 0.37$ for bosons can only be determined numerically by a genuine many-body calculation [5] [7]. Remarkably, the transition from a liquid to a gaseous ground state at $\Lambda_{\mathrm{dB}}^{c} \simeq 0.68$ [8], where also the critical point for a system of bosons disappears, is continuous. Moreover, its location is fixed by a vanishing scattering length, i.e. by two-body physics. Indeed, as noted by Lieb [58], a necessary condition for a gaseous ground state is that the two-body interaction $V(\boldsymbol{x})$ in Eq. (2) has no bound state and a positive scattering length. In the following, we will argue that for interactions considered here in connection with Eq. (3), this condition is also sufficient. Moreover, the liquid and gaseous ground states are separated by a quantum tricritical point. Specifically, we follow an approach due to Sachdev [59] and consider the transition out of the vacuum state into one with a finite particle density n as a function of the chemical potential μ. In the case where the ground state is a gas, the associated effective field theory is the well known ψ^4-theory for a complex scalar field. In a formal manner, this can be derived by starting from the microscopic action of a Bose system with pure two-body interactions as described by Eq. (2). The associated generating functional $Z[J] = \int D\psi \exp\left(-S[\psi]/\hbar + \int J\psi\right)$ for the correlation functions of the complex scalar field $\psi(\tau, \boldsymbol{x})$ can be written as a functional integral with action

$$S[\psi] = \int_\tau \int_{\boldsymbol{x}} \left\{ \psi^*(\tau,\boldsymbol{x})(\hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu)\psi(\tau,\boldsymbol{x}) + \frac{1}{2}|\psi(\tau,\boldsymbol{x})|^2 \int_{\boldsymbol{x}'} V(\boldsymbol{x}-\boldsymbol{x}')\,|\psi(\tau,\boldsymbol{x}')|^2 \right\} . \tag{26}$$

At the mean-field level, the effective potential for field configurations with no dependence on the time and spatial variables τ and $\boldsymbol{x}$, where $|\psi|^2 = n$ can be identified with the particle density, has the form $V_{\mathrm{eff}}^{(0)} = -\mu n + (g/2)n^2$. The coefficient $g = 4\pi\hbar^2 a/m > 0$ is fixed by the two-body scattering length in vacuum. More precisely, as mentioned above in the context of point c) on the exactness of the Bogoliubov approach at low energies, in the naive mean-field approach $g \to g^{(0)}$ contains the scattering length only in the Born approximation, which is ill-defined for potentials which increase more strongly than $1/r^3$ at short distances. This problem is eliminated in the formulation based on an effective potential in Eq. (29) below. Provided that $g > 0$, the onset transition from the vacuum to a superfluid gas to lowest order in the density is properly accounted for in terms of a mean-field description. In particular, the density of bosons $n(\mu) = \mu/g + \ldots$ rises linearly for $\mu \to 0^+$ while $n(\mu) \equiv 0$ vanishes for negative values of the chemical potential. Thus, $\mu = 0, g > 0$ is a line of quantum critical points which separates the vacuum state from a superfluid gas at finite density [59]. Despite the finite jump in the compressibility from $\tilde{\kappa} = \partial n/\partial\mu = 0$ to $\tilde{\kappa} = 1/g > 0$, the vacuum to superfluid transition is a continuous one. Indeed, approaching the line $\mu = 0$ from above, the correlation length is the well known healing length $\xi = \hbar/\sqrt{2m\mu} = (8\pi n a)^{-1/2}$ of a weakly interacting BEC which diverges as $\mu \to 0^+$. Moreover, using the zero temperature Gibbs-Duhem relation $\mu = u + p/n$ which connects the chemical potential and the pressure to the energy u per particle, both $u(n) \to gn/2 = \sqrt{gp/2}$ and the density $n(p) \to \sqrt{2p/g}$ vanish in the zero pressure limit, as required for a gas.

[7] In the case of fermions $\Lambda_{\mathrm{dB}}^{\mathrm{c,solid}}|_F \simeq 0.42$ is substantially larger because fermions prefer to stay localized near a discrete set of lattice sites even for larger values of the zero point motion. The ground state of ^{3}He at zero pressure is a liquid since its de Boer parameter $\Lambda_{\mathrm{dB}} \simeq 0.45$ lies above this critical value.

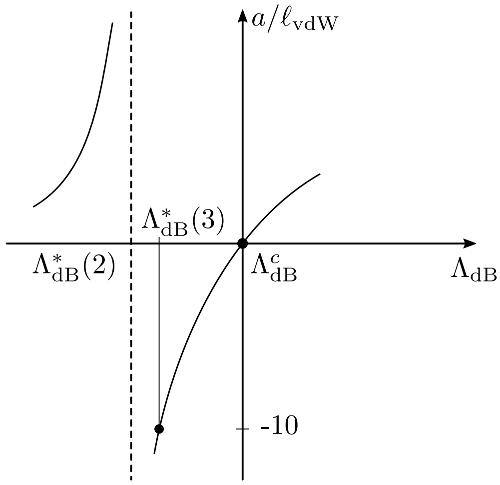


FIG. 6: Qualitative dependence of the scattering length in units of the van der Waals length ℓ_{vdW} as a function of the de Boer parameter defined in Eq. (4). The last two-body bound state disappears beyond the pole of the scattering length at $\Lambda^*_{\mathrm{dB}}(N=2)$ indicated by the dashed vertical line. The scattering length reaches zero at a critical value $\Lambda^c_{\mathrm{dB}} \simeq 0.68$, beyond which it stays positive. The value $\Lambda^*_{\mathrm{dB}}(N=3)$ for the disappearance of three-body bound states is also indicated.

The range of de Boer parameters where a given microscopic interaction gives rise to a positive scattering length and thus a gaseous ground state is determined by the solution of the two-body problem. In the regime $\Lambda_{\mathrm{dB}} \ll 1$, there is a large number $N_b \simeq 1/(\pi\Lambda_{\mathrm{dB}}) \gg 1$ of s-wave bound states. Upon reduction of the strength of the attractive interaction, their number decreases and eventually reaches zero at a critical value of the de Boer parameter. In physical terms, this happens when the van der Waals length $\ell_{\mathrm{vdW}} = (mC_6/\hbar^2)^{1/4}/2$ has decreased to a value of the order of the short distance scale σ. For the specific case of a Lennard-Jones potential, the limit beyond which the two-body Hamiltonian $\hat{H}_2$ no longer has a bound state is reached at $\Lambda^*_{\mathrm{dB}}(N=2) = 0.423\ldots$ or $\ell_{\mathrm{vdW}} = 1.09\,\sigma$. At this point, the scattering length jumps form $+\infty$ to $-\infty$, as sketched in Fig. 6. In fact, this is close to the situation present in ^{4}He, where $\Lambda_{\mathrm{dB}} \simeq 0.42$ and the attractive part of the two-body interaction is just barely sufficient to give rise to a bound state with a binding energy $B_2 \simeq k_B \cdot 1.7\,\mathrm{mK}$. Upon further increasing the de Boer parameter, the scattering length increases monotonically from $-\infty$ towards zero, which is reached at some critical value Λ^c_{dB}. Specifically, one finds $\Lambda^c_{\mathrm{dB}} = 0.679\ldots$ for a Lennard-Jones potential, corresponding to a van der Waals length $\ell_{\mathrm{vdW}}|_c \simeq 0.86\,\sigma$. Increasing Λ_{dB} beyond its critical value, the scattering length stays positive. In particular, near Λ^c_{dB}, the scattering length

$$a(\Lambda_{\mathrm{dB}}) = a_\Lambda\,\ell_{\mathrm{vdW}}\left(\Lambda_{\mathrm{dB}} - \Lambda^c_{\mathrm{dB}}\right) + \ldots \tag{27}$$

vanishes linearly with a positive numerical constant a_Λ of order one. The regime $g > 0$ of a gaseous ground state is realized for $\Lambda_{\mathrm{dB}} > \Lambda^c_{\mathrm{dB}}$. As mentioned above, the same situation applies for ultracold gases despite $\Lambda_{\mathrm{dB}} \ll 1$ provided the scattering length is positive and the many two-body bound states are inaccessible on relevant time scales.

For negative scattering lengths, the ground state of a uniform Bose fluid is obviously not a gas. As will be shown below, there is a finite range of the Boer parameters below Λ^c_{dB}, where the ground state is a liquid which is stabilized by repulsive three-body interactions. Its properties near the first-order transition to the vacuum state are determined by a solution of the three-body problem. Now, as predicted by Efimov [60] in a nuclear physics context, identical bosons support three-body bound states in a regime where the scattering length is negative and no two-body bound state exists. As indicated in Fig. 6, where the critical value $\Lambda^*_{\mathrm{dB}}(N=3) \simeq 0.45$ for the disappearance of the last three-body bound state is shown, this requires a minimum value of the magnitude $|a_-(3)|$ of the associated two-body scattering length which is $a_-(3) = -9.6\,\ell_{\mathrm{vdW}}$ for a Lennard-Jones interaction [61]. The three-body bound states predicted by Efimov were first observed in an ultracold gas of ^{133}Cs by Kraemer et al. [62]. Surprisingly, the ratio $|a_-(3)|/\ell_{\mathrm{vdW}} \simeq 8-10$ turned out to vary in an only narrow range for many different atoms [63]. An explanation for this so called van der Waals universality has been given independently by Wang et al. [64] and by Schmidt et al. [65]. Wang et al. consider direct two-body interactions with different single channel potentials at short distance but identical van der Waals tails. The solution of the associated three-body problem then shows that the ratio $(a_-(3)/\ell_{\mathrm{vdW}})|_{N_b\gg 1} = -9.45$ approaches a universal value in the limit of a large number $N_b \gg 1$ of bound states [64]. In practice, a change in the scattering length relies on the use of Feshbach resonances. As shown by Schmidt et al. [65] within a standard two-channel model, a nearly universal value of the ratio $a_-(3)/\ell_{\mathrm{vdW}} \simeq -9$ then appears only in the open-channel dominated limit $s_{\mathrm{res}} \gg 1$ [66]. Moreover, considerable deviations towards more negative numbers were predicted for Feshbach resonances with intermediate strength $s_{\mathrm{res}} \simeq 1$. They have recently been observed in ^{39}K by the JILA group, see Chapurin et al. [67] and Xie et al. [68].

For the many-body problem at finite density, the endpoint at $g = 0$ of the line $\mu \equiv 0$ turns out to be a quantum tricritical point (see Fig. 7). It separates the continuous onset transition from the vacuum to a gaseous state in the regime $g > 0$ from a first-order transition at $\mu_c < 0$ between the vacuum and a finite density liquid for negative values of the scattering length. In order to properly deal with the regime $g < 0$, it is necessary to include the quantum fluctuations of the field $\psi(\tau, \boldsymbol{x})$ to all orders. On a formal level, this can be expressed in terms of an effective potential

$$\Gamma[\psi] = \sum_{N=1}^{\infty} \frac{1}{N!} \int_{p_1 \ldots q_N} \Gamma_N(p_1 \ldots p_N\, q_1 \ldots q_N)\, \psi^*(p_1) \ldots \psi^*(p_N)\, \psi(q_1) \ldots \psi(q_N) = \int_{\tau, \boldsymbol{x}} \left\{ V_{\text{eff}}[\psi] + \psi^* \tilde{D} \psi + \ldots \right\} \tag{28}$$

which is defined via a Legendre transform $\Gamma[\psi] = \ln\{Z[J]/Z[0]\} - \int J\psi$ of the generating functional $Z[J]$ associated with the action (26) [8]. In practice, the Legendre transform can only be performed if one is able to determine the expectation value of the field for an arbitrary form of external source $J(\tau, \boldsymbol{x})$ and then invert this relation to determine $J(\tau, \boldsymbol{x})$ as a functional of the associated configuration $\psi(\tau, \boldsymbol{x})$. The resulting exact vertex functions Γ_N are essentially the amplitudes for scattering processes with N incoming and N outgoing particles. Knowledge of the Γ_N, including their dependence on the $2N$ momentum variables $p_1 \ldots q_N$ which are constrained only by translation invariance in space and time $p_1 + \ldots + p_N = q_1 + \cdots + q_N$, therefore requires a complete solution of the N-body problem. This is clearly impossible. Fortunately, however, for the discussion of the behavior near the quantum tricritical point, which is a zero density fixed point, we need only the leading non-vanishing terms in the expansion of the effective potential

$$V_{\text{eff}}[\psi] = -\mu|\psi|^2 + \frac{g}{2}\,|\psi|^4 + \frac{\lambda_3}{3}|\psi|^6 + \ldots \tag{29}$$

associated with a time and space independent 'classical' field ψ. Here, as mentioned above, the prefactor $g = \Gamma_2(0) = 4\pi\hbar^2 a/m$ of the quartic term is fixed by the exact value a of the two-body scattering length which may be defined through the asymptotic behavior $\psi_{E=0}(\boldsymbol{x}_1, \boldsymbol{x}_2) = 1 - a/r_{12}$ of the two-body wave function at zero energy. If g is positive, the transition out of the vacuum state is completely fixed by the first two terms in Eq. (29), recovering the scenario for a gaseous ground state discussed above. For negative g, in turn, one needs the next-to-leading contribution $\sim |\psi|^6$. Its prefactor $\lambda_3 = \hbar^2 D/2m$ arises from the zero momentum limit $\Gamma_3(0) = \hbar^2 D/m$ of the vertex function which is associated with effective three-body interactions. The corresponding parameter D has been called the three-body scattering hypervolume by Tan [70]. It has dimension $(\text{length})^4$ and may be defined by the asymptotic behavior

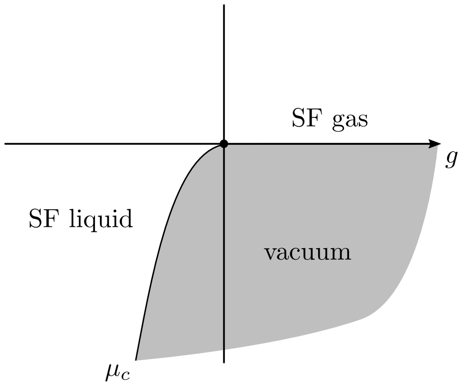
$$\psi_{E=0}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)|_{a=0} = 1 - \frac{\sqrt{3}\, D}{2\pi^3 (r_{12}^2 + r_{13}^2 + r_{23}^2)^2} + \ldots \tag{30}$$

of the three-body wave function at zero energy and vanishing scattering length [70]. Similar to the standard connection between two-body bound states and poles of the scattering length, the occurence of three particle bound states is determined by poles of the hypervolume D. Now, as indicated in Fig. 6, the last three-body bound state disappears at a finite negative scattering length $a_-(3) \simeq -9\,\ell_{\text{vdW}}$. Near Λ^c_{dB}, therefore, the vertex function Γ_3 has no poles. Moreover, the associated hypervolume $D(a\!=\!0) > 0$ is positive near the zero of the scattering length at Λ^c_{dB} according to a numerical solution of the three-body problem with a Lennard-Jones interaction [71]. This implies a repulsive effective three-body force and an energy per particle $u(n)|_{a=0} = (\hbar^2 D/6m) \cdot n^2$ which scales quadratically with density n [70]. At vanishing scattering length, therefore, the many-body Bose fluid is stabilized by repulsive three-body interactions, a behavior quite different from that of the naively expected ideal Bose gas. In particular, the finite density fluid at $a = 0$ is characterized by a non-trivial relation between pressure and chemical potential of the form

$$p(\mu)|_{a=0} = \left(\frac{8m}{9\hbar^2 D}\right)^{1/2} \cdot \mu^{3/2} \quad \rightarrow \quad \mu(n)|_{a=0} = \frac{\hbar^2 D}{2m} \cdot n^2\,. \tag{31}$$

As a result, the density $n(\mu) = \partial p/\partial\mu$ scales with the square root of the chemical potential rather than the linear behavior found for positive scattering lengths. This is a consequence of the non-standard critical exponent $\beta = 1/4$ associated with the appearance of a finite order parameter $|\psi|(\mu) \sim \mu^{\beta}$ right at the quantum tricritical point which separates the gaseous from the liquid ground state in the zero density limit.

[8] For an introduction to the formalism see e.g. the book by Zee [69]. Due to Galilei invariance, derivatives only appear in the covariant form $\tilde{D} = \hbar\partial_\tau - \hbar^2\nabla^2/2m$.



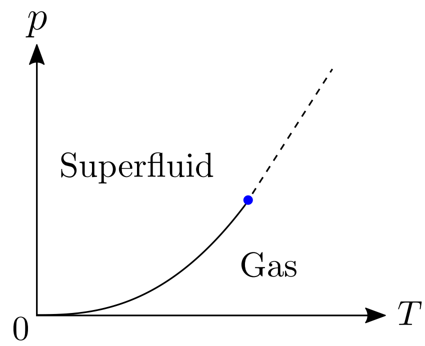


FIG. 7: Zero temperature phase diagram as a function of the chemical potential μ and the deviation $g \sim \Lambda_{\rm dB} - \Lambda^c_{\rm dB}$ of the de Boer parameter from its critical value. The gaseous ground state in the regime $g > 0$ arises from the vacuum at $\mu < 0$ via a continuous transition. For $g < 0$, the ground state is a liquid. It is separated from the vacuum by a first-order transition at $\mu_c < 0$. The point $\mu = g = 0$ is a quantum tricritical point. The finite temperature phase diagram for $\Lambda_{\rm dB} < \Lambda^c_{\rm dB}$ on the right is adapted from Son et al. [57]. Beyond a tricritical point at $T^* \simeq \hbar^2\bar{n}^{2/3}/m$, the transition from a superfluid liquid to a non-superfluid gas changes from first order to a continuous one.

In the regime $g < 0$, the symmetry broken phase with a finite density $n(\mu) = |\bar{\psi}|^2 \neq 0$ appears already beyond a negative value

$$\mu_c = -3g^2/(16\lambda_3) = -6\pi^2\,\hbar^2 a^2/(mD) \tag{32}$$

of the chemical potential, which vanishes with the square of the distance from the quantum tricritical point as indicated in Fig. 7. By the Gibbs-Duhem relation, the critical chemical potential $\mu_c = u(p = 0)$ coincides with the energy per particle since the pressure vanishes along the line separating the vacuum from the finite density liquid. Right on the line $\mu = \mu_c$, the density jumps from zero in the vacuum state $\mu < \mu_c$ to a finite value

$$\bar{n} = n(\mu_c) = 3|g|/(4\lambda_3) = 6\pi\,|a|/D \quad\rightarrow\quad \bar{n}\sigma^3 = 6\pi\,|a|\sigma^3/D \underset{\rm LJ}{\longrightarrow} 1.32\,(\Lambda^c_{\rm dB} - \Lambda_{\rm dB}) + \ldots\,. \tag{33}$$

The dimensionless product $\bar{n}\sigma^3$ therefore approaches zero linearly with the deviation from the quantum tricritical point. The numerical prefactor in the final expression is specific for a Lennard-Jones interaction, where the factor $a_\Lambda = 3.828$ in Eq (27) and the three-body hypervolume $D(a\!=\!0) = (86 \pm 2)\,\ell^4_{\rm vdW}$ near the last zero crossing of the scattering length have been determined by Mestrom et al. [71]. Despite the considerable deviation $\Lambda^c_{\rm dB} - \Lambda_{\rm dB} \simeq 0.26$ of the de Boer parameter of ^{4}He from the critical value for a liquid-gas transition, a naive application of Eq (33) predicts a dimensionless density $\bar{n}\sigma^3 \simeq 0.34$ for ^{4}He at zero pressure which is close to the observed value. This agreement is again a fortuitous coincidence, however, because two ^{4}He atoms form a weakly bound dimer and thus the relation (27) does not apply. A system rather close to the quantum tricritical point, still on the liquid side, would be ^{2}He. Its de Boer parameter is expected to be $\Lambda_{\rm dB} \simeq \sqrt{2} \cdot 0.42 = 0.59$ due to the factor two in mass. Unfortunately, this extremely dilute superfluid liquid does not exist in nature because the di-proton is not bound [9]. The evolution of the finite temperature phase diagram in the regime of de Boer parameters between $\Lambda_{\rm db} \simeq 0.37$ and $\Lambda^c_{\rm dB}$, where the ground state at vanishing pressure is a liquid, has been discussed by Son et al [57]. Surprisingly, this diagram is of the familiar form observed in ^{4}He (see Fig. 2) only in a finite range of $\Lambda_{\rm dB}$ above 0.37. For values that correspond to the hypothetical ^{2}He fluid and up to $\Lambda^c_{\rm dB}$, in turn, the critical endpoint of the λ-line on the liquid-gas boundary has disappeared. Instead, as shown in Fig. 7, there is a tricritical point along the coexistence line between a superfluid liquid and the normal gas above which the transition changes from being first order to a continuous one. Its temperature $T^* \simeq \hbar^2\bar{n}^{2/3}/m$ is set by the finite density $\bar{n}$ of the liquid ground state at zero pressure given in Eq (33) which also determines the jump in density below T^* by the simple relation $\Delta n = [1 - (T/T^*)^{3/2}]\,\bar{n}$ [57]. Since $\bar{n} \to 0$ in the limit of vanishing scattering length, the tricritical point shifts to zero temperature and then coincides with the quantum tricritical point $\mu = g = 0$ shown in Fig. 7 on the left.

Regarding a possible realization of a liquid state in ultracold Bose gases near vanishing scattering length which is stabilized by repulsive three-body interactions, it is necessary to account for the finite imginary part of the three-body scattering hypervolume that is present at generic zero crossings of a in the standard regime where the de Boer parameter $\Lambda_{\rm dB}$ is much less than one. As will be discussed in Lecture III, this leads to a corresponding loss rate $\Gamma_3 = -\hbar\,{\rm Im}(D)\,n^2/m$ [72]. Experimentally, these losses

[9] For a discussion of the thermodynamics and life time of stars if a di-proton bound state would exist, see L. A. Barnes, arXiv:1512.06090 [astro-ph.SR].

have been studied by Shotan et al. [73], who measured the recombination length L_m defined by $\mathrm{Im}\, D \simeq L_m^4$ near a zero crossing of the scattering length at $B \simeq 850\,\mathrm{G}$ in ^{7}Li. Remarkably, the observed value $L_m \simeq 4\,\ell_{\mathrm{vdW}}$ is close to that quoted above for the fourth root $D^{1/4} \simeq 3.1\,\ell_{\mathrm{vdW}}$ of the purely real three-body scattering hypervolume near the zero crossing of the scattering length at Λ_{dB}^c, where no two-body bound state exists. Now, according to Eq. (33), the density of a liquid state stabilized by three-body repulsion is of order $\bar{n}\ell_{\mathrm{vdW}}^3 \simeq |a|/\ell_{\mathrm{vdW}}$ for typical values $\mathrm{Re}\, D \simeq (\ell_{\mathrm{vdW}})^4$. In practice, such high densities are not accessible with ultracold gases. However, as suggested by Petrov [74], a dilute liquid phase of bosons at negative scattering length which is stabilized by repulsive three-body interactions might be realized in a situation where two internal states $|\uparrow\rangle$ and $|\downarrow\rangle$ are coupled by an rf-field. By varying the effective Rabi coupling, the scattering length in the symmetric configuration $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ can be tuned to zero. The associated three-body scattering hypervolume $D(a=0) \simeq a_{\uparrow\uparrow}^4/\xi$ is large and positive provided $\xi = (a_{\uparrow\downarrow} + a_{\uparrow\uparrow})/(a_{\uparrow\downarrow} - a_{\uparrow\uparrow}) \ll 1$. In particular, it is a factor $1/\xi \gg 1$ larger than the characteristic magnitude $\mathrm{Im}\, D \simeq a_{\uparrow\uparrow}^4$ of its imaginary part, as determined by the standard scaling of the three-body loss rate. Neglecting losses, the resulting effective potential (29) gives rise to a dilute Bose liquid in the regime where $a < 0$. Its dimensionless density $\bar{n}a_{\uparrow\uparrow}^3 \simeq \xi\,|a|/a_{\uparrow\uparrow}$ vanishes linearly with the scattering length as in Eq. (33) and - moreover - is small enough to be accessible with dilute ultracold gases. The state is a three-body interaction analog of self-bound droplets in two - component Bose gases which are stabilized by the Lee-Huang-Yang contribution to the interaction energy. They were predicted by Petrov [75] and have been realized experimentally by Cabrera et al. [76]. In fact, liquid-like droplets of bosons have been observed earlier by Ferrier-Barbut et al. [77] in dipolar gases, where the mean-field instability due to the attractive part of the dipolar interaction is eliminated by the repulsive LHY - correction $e(n)|_{\mathrm{LHY}} \sim gn^2\,(na^3)^{1/2}$ to the ground state energy density. This stabilizes droplets at densities of order $10^{14}\,\mathrm{cm}^{-3}$ [77].

Self-bound droplets and N-body bound states near vanishing scattering length In the regime $\Lambda_{\mathrm{dB}} < \Lambda_{\mathrm{dB}}^c$ of negative scattering lengths, the ground state at vanishing pressure is a superfluid liquid. By the Gibbs-Duhem relation, the energy per particle $u(p=0) = \mu_c < 0$ is negative. A given number N of particles thus has an extensive binding energy $B_N = |u(p=0)|\,N$. Moreover, since the liquid has a finite density $\bar{n}$ at zero pressure, the radius of an N-cluster scales like $R_N \simeq (N/\bar{n})^{1/3}$. In the limit where the scattering length approaches zero, both $u(p=0)$ and $\bar{n}$ vanish. The zero pressure liquid thus evaporates into a gas precisely at the quantum tricritical point $\mu = g = 0$. This is true, however, only in the thermodynamic limit. For finite particle numbers, the binding energy B_N is reduced because particles on the surface are less bound than those in the bulk. For the specific case of a Lennard-Jones interaction, this has been studied numerically for small clusters by Meierovich et al. [78] and by Sevryuk et al. [79]. In particular, it has been found that, at finite N, quantum unbinding appears at values $\Lambda_{\mathrm{dB}}^*(N) < \Lambda_{\mathrm{dB}}^c = 0.679...$ of the de Boer parameter which are considerably lower than what is expected in the thermodynamic limit. This observation can be understood by including a finite, positive surface energy f_s per particle in the liquid phase, which also accounts for the essentially flat radial density distributions found numerically near Λ_{dB}^c [79]. The surface energy is defined by the subleading term in the expansion

$$E_0(N) = u\,N + f_s\,N^{2/3} + \ldots \tag{34}$$

of the N-body ground state energy for $N \gg 1$. Taking into account the surface contribution, the condition $E_0(N+1) = E_0(N)$ for the unbinding of an N-cluster, which is equivalent to a vanishing single particle addition energy $\mu(N) = E_0(N+1) - E_0(N) = 0$, can be written in the form

$$\frac{-3\,u}{2f_s}\left[\Lambda_{\mathrm{dB}}^*(N)\right] = N^{-1/3}\,. \tag{35}$$

The finite size scaling of the deviation $\Lambda_{\mathrm{dB}}^c - \Lambda_{\mathrm{dB}}^*(N)$ for $N \gg 1$ is thus determined by the dependence of the bulk energy u and the surface energy f_s per particle on the de Boer parameter. Now, Eq. (32) shows that the energy per particle $u(p=0) = \mu_c$ on the zero pressure line separating the vacuum from the finite density liquid vanishes quadratically with the distance from the quantum tricritical point. To determine how the surface energy f_s per particle vanishes near Λ_{dB}^c, we use the result for the underlying surface tension [10]

$$\bar{\sigma} = \frac{\lambda_3 \bar{n}^3}{6\,\kappa_0} \simeq \frac{\hbar^2\,a^2}{m\,D^{3/2}} \sim (\Lambda_{\mathrm{dB}}^c - \Lambda_{\mathrm{dB}})^2 \tag{36}$$

derived by Bulgac [81] on the basis of the exact domain wall solution $n(z) = \bar{n}/\left(1 + \exp(2\kappa_0 z)\right)$ for the liquid-to-vacuum boundary with an effective potential of the form (29) right at the critical value (32) of the chemical potential. The associated healing

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[10] We use a bar in the surface tension $\bar{\sigma}$ to distinguish it from the short distance length scale σ. Note also that the exponent $\nu_u = \nu_t/\phi_t = 1$ for the divergence of the correlation length $1/\kappa_0$ along the first-order transition line $\mu = \mu_c$ is a subsidiary tricritical exponent in the notation of Griffiths [80]. The relevant crossover exponent $\phi_t = 1/2$ is determined by the quadratic behavior (32) of the chemical potential near the quantum tricritical point.

length $1/\kappa_0 = \hbar/\sqrt{2m|\mu_c|} \simeq \sqrt{D}/|a|$ diverges linearly with the distance from the quantum tricritical point, implying that the surface tension vanishes quadratically. This is consistent with a scaling relation due to Widom [82] which connects the exponent of the surface tension $\bar{\sigma} \sim 1/\xi^{d-1}$ with that of the correlation length. More precisely, the scaling argument by Widom states that $\bar{\sigma} \simeq k_B T_c/\xi^{d-1}$ vanishes like the characteristic energy $k_B T_c$ at a finite temperature phase transition divided by the surface area ξ^{d-1} of a domain with size ξ. For the phase transition at the quantum tricritical point studied here, the role of $k_B T_c$ is apparently played by $\hbar^2/(m\sqrt{D})$. Combining the results (33) for the average interparticle spacing $\bar{n}^{-1/3}$ and Eq. (36) for the surface tension shows that the surface energy $f_s \simeq 4\pi\,\bar{n}^{-2/3}\cdot\bar{\sigma} \sim |\Lambda_{\rm dB} - \Lambda^c_{\rm dB}|^{4/3}$ vanishes with a non-trivial exponent 4/3 near the quantum tricritical point. Based on Eq. (35), the threshold values $\Lambda^*_{\rm dB}(N)$ of the de Boer parameter beyond which N-body bound states disappear therefore approach the critical value $\Lambda^c_{\rm dB}$ of the bulk liquid-gas transition according to $\Lambda^c_{\rm dB} - \Lambda^*_{\rm dB}(N) \sim N^{-1/2}$. Moreover, in view of Eq. (27), this leads immediately to a power law behavior

$$-\,a_-(N \gg 1) \simeq \left(\sqrt{D}/N\right)^{1/2} \quad \text{or} \quad N^*(a) \simeq \sqrt{D}/a^2 \tag{37}$$

of the associated scattering lengths $a_-(N)$ or the critical number $N^*(a)$ where self-bound droplets of $N^*(a)$ bosons unbind at a given negative scattering length a. It has the remarkable feature that the three-body scattering hypervolume $D(a=0)$ at vanishing scattering length sets the scale for the unbinding of N-body bound states in the asymptotic limit $N \gg 1$. This is a consequence of the fact that D appears in the leading term $\sim D\,|\psi|^6$ in Eq. (29) which stabilizes the superfluid at both vanishing and small negative scattering lengths, while higher order contributions are negligible near the quantum tricritical point, where $\bar{n} \to 0$.

The result (37) provides a solution to a long standing problem on how to connect well known results in few-body physics to the many-body limit $N \gg 1$. As mentioned above, the existence of three-body bound states for identical bosons had been predicted in the early seventies by Efimov [60]. Many-body bound states exist also for larger particle numbers. This has been studied in detail for $N = 4$, where theory predicts an infinite sequence of two tetramer states per Efimov trimer [83–86]. Experimentally, the lowest tetramer state has been observed by Ferlaino et al. [87] at $a_-(4) \simeq 0.47\,a_-(3)$ and even signatures of a five-body bound state have been inferred from a characteristic feature in the recombination rate of Cesium near a scattering length $a_-(5) \simeq 0.64\,a_-(4)$ [88]. More generally, the energetically lowest N-body bound states, which are the true ground states of the N-particle system in the regime $\Lambda^*_{\rm dB}(N=2) \le \Lambda_{\rm dB} < \Lambda^c_{\rm dB}$, detach from the continuum at a sequence $a_-(N) < 0$ of scattering lengths which apparently approaches zero in a monotonic manner. This has been investigated by von Stecher via numerical solutions of the Schrödinger equation up to $N = 13$ [89]. In particular, it turns out that the consecutive ratios $a_-(4)/a_-(3) \simeq 0.44$, $a_-(5)/a_-(4) \simeq 0.64$ and $a_-(6)/a_-(5) \simeq 0.73$ are not very sensitive to the detailed form of the two-body interactions [90]. An obvious question is then whether the sequence of N-body bound states continues up to $N = \infty$ and - if so - what is the asymptotic scaling of the scattering lengths $a_-(N)$ where they first appear, starting from $a = 0^-$. The finite size scaling theory for self-bound liquid droplets near the quantum tricritical point developed above provides an explicit answer to this in the limit $N \gg 1$. In particular, it shows that the effective binding energy of N-clusters of identical bosons vanishes at a sequence of negative scattering lengths $a_-(N)$ which approach zero in an algebraic fashion as described by Eq. (37). The existence of an infinite sequence of N-body bound states with an accumulation point at $a = 0$ is consistent with a theorem due to Seiringer [91], which states that some N-body bound state must exist for arbitrary small negative scattering lengths. It is also consistent with an earlier theorem by Amado and Greenwood [92] which shows that the number of N-body bound states is finite for any $N \ge 4$ precisely at the position $a_-(N-1)$ where a zero-energy $N-1$-body bound state appears. An experimental verification of the prediction (37) is an open challenge and requires to determine the size dependence in the unbinding of self-bound droplets near the limit $a \to 0^-$ of their stability. Remarkably, a related problem appears in nuclear physics where the binding energy of nuclei with an equal and even number of protons and neutrons depends on the strength of the effective interaction between two alpha particles. Similar to a change of the de Boer parameter discussed above, this interaction may be tuned to a quantum tricritical point which separates a nuclear liquid and an unbound gas of alpha particles [93].

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