Scale and conformal invariance in cold gases

Collège de France, October 27th, 2021

• S-inv in quantum mechanics and statistical physics

Hydrogen atom in 4D $\hat{H} = -\nabla^2 - \frac{2}{\tilde{a}_0 \, r^2}$ obeys

$\hat{H}(b\mathbf{x}) = \hat{H}/b^2 \;\; \rightarrow$ spectrum is continuous !

for $\tilde{a}_0 = 2\pi^2\hbar^2/me_4^2 < 2$ one needs a finite proton radius

an electron binds to a polar molecule only if $d_{\rm el} \gtrsim 2$ Debye

correlation of an order parameter right at a critical point

$$C(x-y) = \langle \phi(x)\phi(y) \rangle \rightarrow \frac{\text{const}}{|x-y|^{d-2+\eta}}$$ i.e. $C(bx) = C(x)/b^{d-2+\eta}$

e.g. superfluids in 2D below $T_{\rm BKT}$ $\langle \hat{\psi}^\dagger(x)\hat{\psi}(y) \rangle \rightarrow n_2 \left(\frac{\xi_h}{|x-y|} \right)^{\eta(T)}$

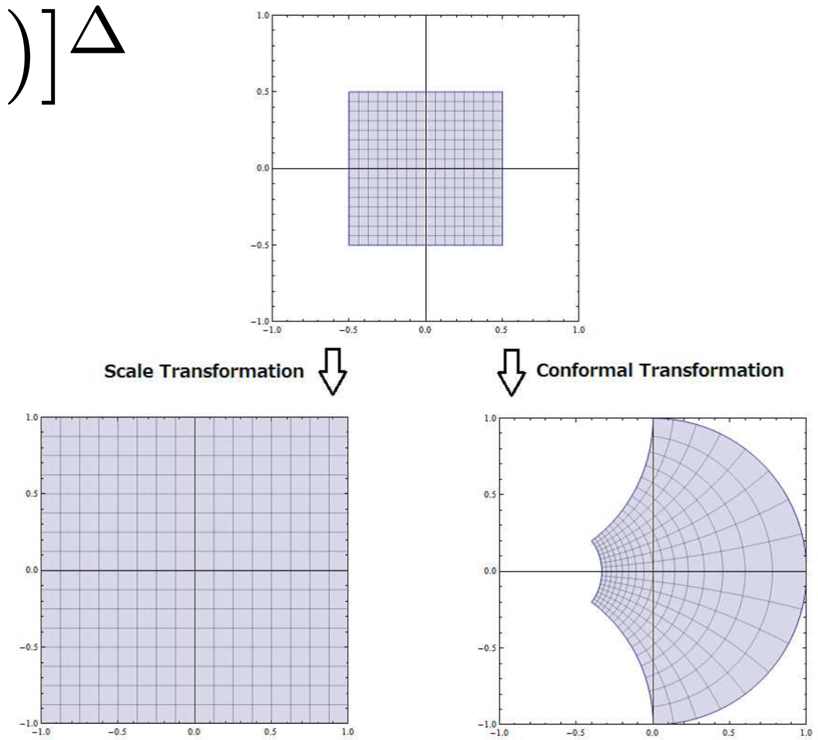
• extending scale to conformal invariance

Weyl transformations of the metric $g'_{\mu\nu}(x') = b^2(x)\delta_{\mu\nu}$

conformal invar. $\langle \hat{\mathcal{O}}(x')\hat{\mathcal{O}}(y')\rangle = \dfrac{\langle \hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\rangle}{[b(x)\, b(y)]^{\Delta}}$

holds even with spatially dependent b(x)

for primary operators $\hat{\mathcal{O}}(x)$



electrodynamics in 3+1 dimension

$$\langle F_{\mu\nu}(x)\, F_{\lambda\sigma}(y)\rangle = \frac{I_{\mu\lambda}(x)\, I_{\nu\sigma}(y) - I_{\nu\lambda}(x)\, I_{\mu\sigma}(y)}{(x-y)^4}$$

Poincaré group is extended by five additional generators

• Outline

- Scale and conformal invariance in non-relativistic many-body physics

  scale invariance necessarily implies conformal invar. which has only one additional generator

- Equation of state and dynamics in scale invariant gases

- Viscosity and (nearly) perfect quantum fluids

• scale invariance and conservation laws

$$\hat{H}(bx_1,\ldots bx_N) = \hat{H}(x_1\ldots x_N)/b^2 \quad \text{implies} \quad i[\hat{H},\hat{D}] = 2\hat{H}$$

generator of dilatations $\hat{D} = \int \mathbf{x}\cdot\hat{\mathbf{g}}(\mathbf{x})/\hbar \to \int \hat{n}(\mathbf{x})\,\mathbf{x}\cdot\nabla\hat{\varphi}(\mathbf{x})$

local conservation law $\partial_t\hat{\rho}_D + \mathrm{div}\,\hat{\mathbf{j}}_D = 0$ obeyed if

$(*)$ $2\,\hat{\varepsilon}(\mathbf{R}) = \hat{\Pi}_{ii}(\mathbf{R})$ (in relativistic theories $T^{\mu}{}_{\mu}(x) \equiv 0$)

trace of stress tensor in many-body phys. Martin/Schwinger 1959

$(*)$ requires $r\,dV(r)/dr = -2V(r) \to$ $V(r) = c_d/r^2$

or $V(r) = \bar{g}_2\delta(\mathbf{x})$

Pitaevskii/Rosch 1998

• conformal invariance in non-relativistic physics

Niederer 1972 symmetries of the free Schroedinger equation

include **expansion** = Galilei transformation with $\mathbf{v}(\mathbf{x}) = d_0\mathbf{x}$

$$(t,\mathbf{x}) \xrightarrow[\Sigma]{} (-1/t, \mathbf{x}/t) \xrightarrow[\text{Trans}]{} (-1/t - d_0, \mathbf{x}/t) \xrightarrow[\Sigma^{-1}]{} \left(\frac{t}{1+d_0 t}, \frac{\mathbf{x}}{1+d_0 t} \right)$$

generated by $\hat{C} = (m/2) \int \mathbf{x}^2 \, \hat{n}(\mathbf{x})$

Galilei group is extended by $\hat{D}$ and $\hat{C}$, together with $\hat{H}$

they form a closed sub-algebra with symmetry SO(2,1)

Hagen 1972 for spinless particles scale invariance implies

conformal invariance, i.e. there is no new conservation law a scale invariant many-body problem in 1D or 3D is obtained by

$$\hat{H}_{\rm unitary} = \hat{H}_0 \quad \text{and} \quad \psi(x_1 \ldots x_N) \in \mathcal{D}(\hat{H}_{\rm unitary}) \quad \text{if}$$

$$\psi(x_1 \ldots x_N) \to \text{const}\, |x_i - x_j|^{2-d} + \text{const}'\, |x_i - x_j|^{4-d} \ \text{ for } \ |x_i - x_j| \to 0$$

Werner / Castin 06 Son / Wingate 06

a list of scale invariant gases in 1D and 3D

unitary quantum gases	Bose	repulsive Fermi	attractive Fermi
one dimension	Tonks-Girardeau gas, $\mu = \varepsilon_F^{\rm eff}$	$\uparrow\downarrow$ - Fermionization, $\mu = 4\,\varepsilon_F$	TG gas of dimers, $\mu_{\rm eff} = \varepsilon_F/4$
three dimensions	stable only for $n\lambda_T^3 \lesssim 1$	unstable repulsive branch of FBR	unitary Fermi gas, $\mu \simeq 0.37\,\varepsilon_F$

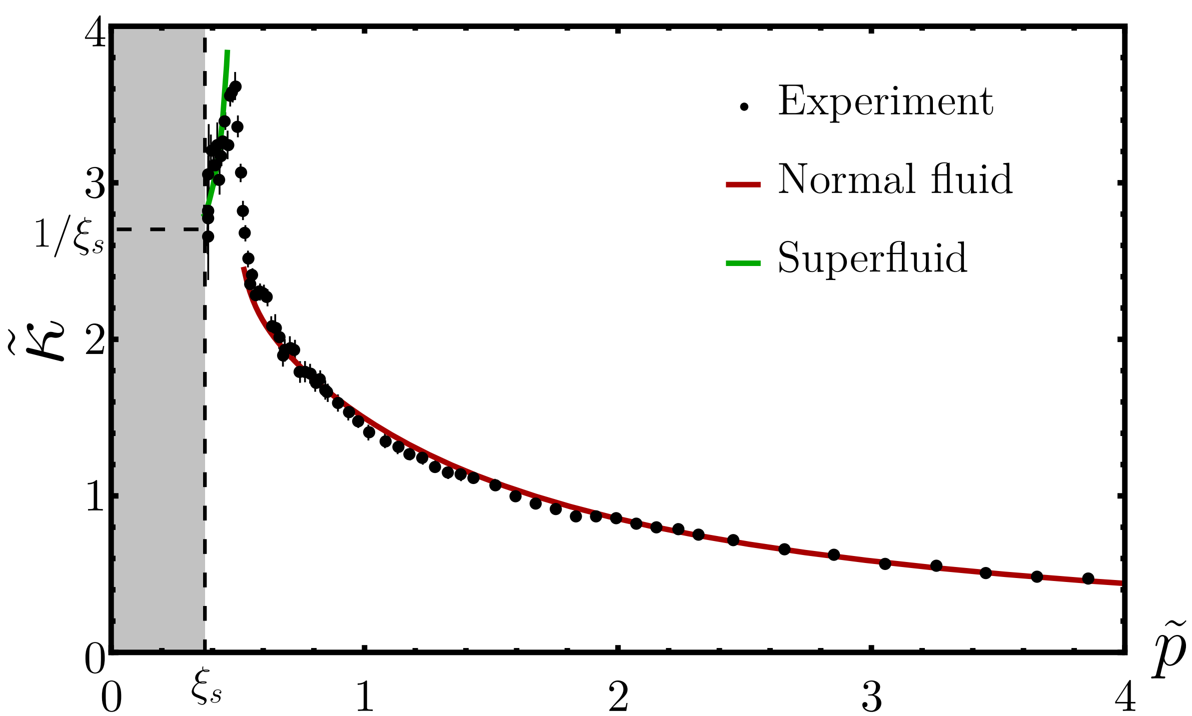
- **universal thermodynamics of scale invariant gases**

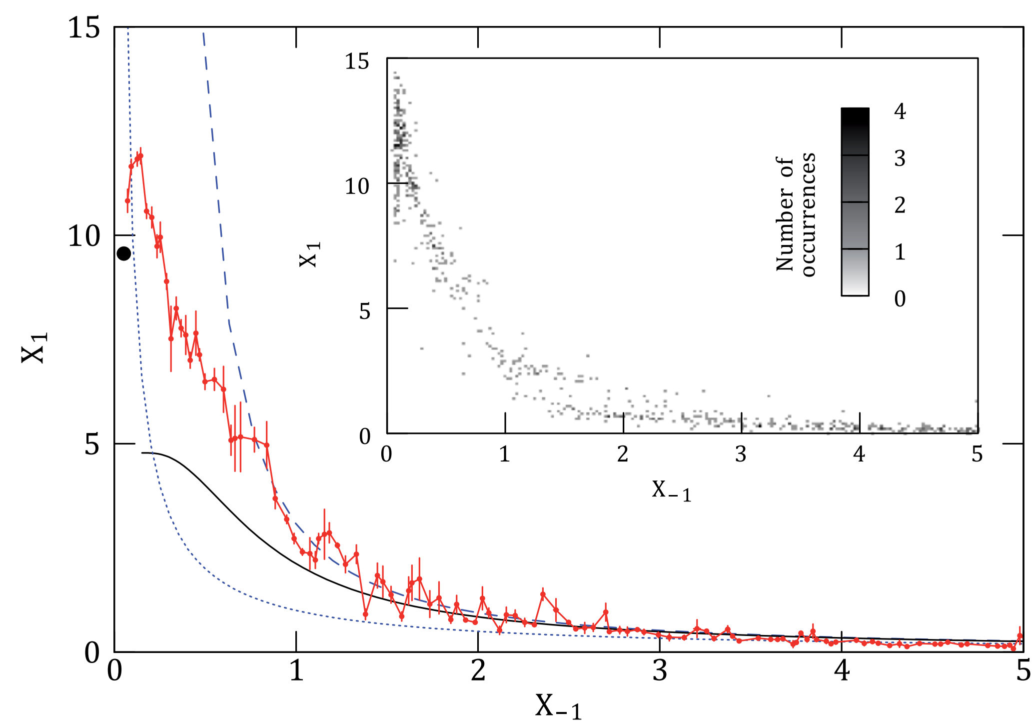
$2\,\hat{\varepsilon} = \hat{\Pi}_{ii}$ implies $2\,U = d \cdot pV$ in thermal equilibrium

$$T\left(\frac{\partial p}{\partial T}\right)_V = p + \left(\frac{\partial U}{\partial V}\right)_T \xrightarrow[\text{s-inv.}]{} \ln\left(\frac{T(\tilde{p}_1)}{T(\tilde{p}_0)}\right) = \frac{2}{d+2}\int_{\tilde{p}_0}^{\tilde{p}_1} \frac{d\tilde{p}}{\tilde{p} - 1/\tilde{\kappa}(\tilde{p})}$$

Planck 1897 determine absolute temperature from EOS

density profiles in cold gases give $n(V_{\text{ext}}) \to p(\mu)$ and $\kappa(\mu)$





• dynamical consequences of scale invariance

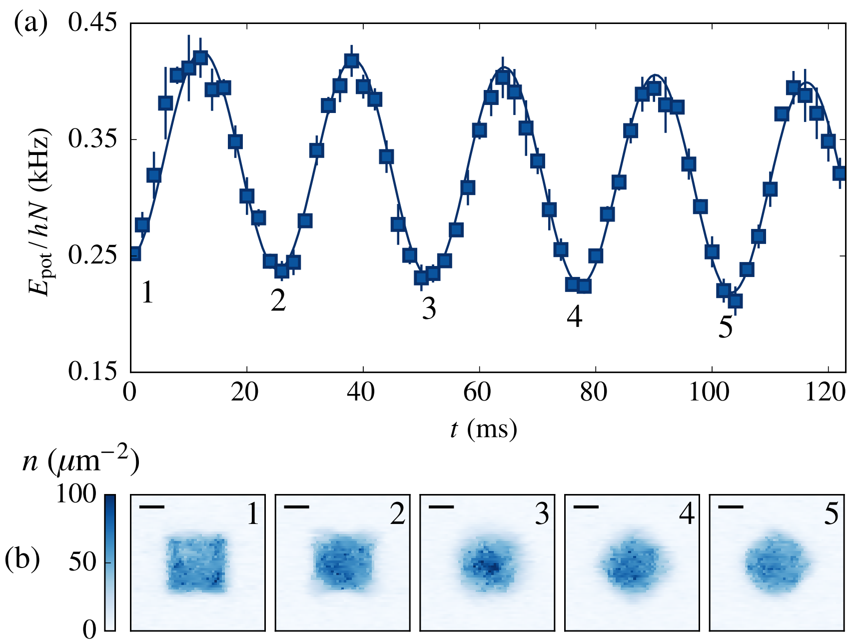
operators $\hat{H}, \hat{C}$ and $\hat{D}$ form a closed algebra $SO(2,1)$

Hamiltonian in an isotropic harmonic trap $\hat{H}_\omega = \hat{H} + \omega^2 \hat{C}$

define $\hat{L}_3 = \hat{H}_\omega/(2\hbar\omega)\,,\ \hat{L}_1 = \hat{L}_3 - \omega\,\hat{C}/\hbar$ and $\hat{L}_2 = \hat{D}/2$

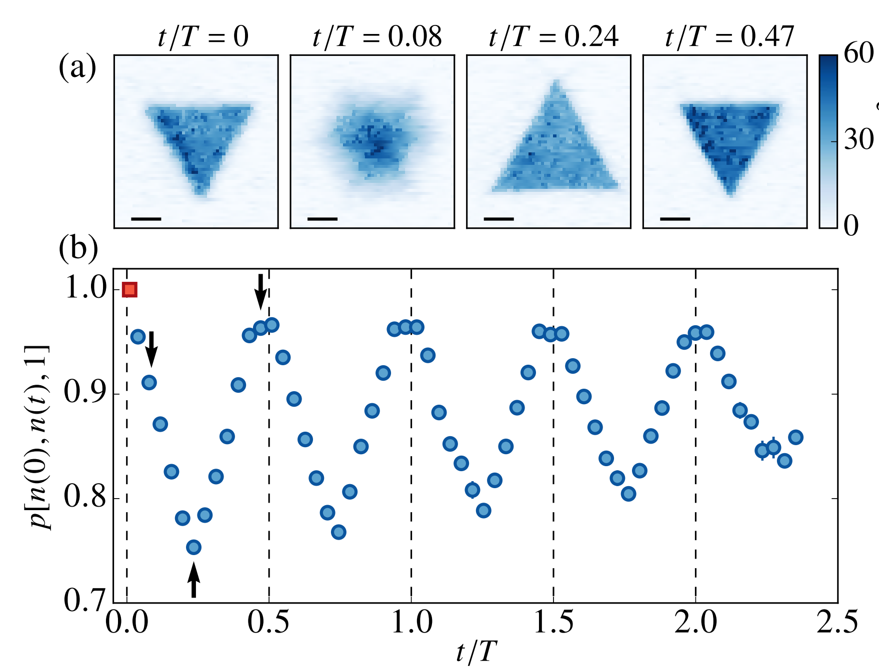
$\rightarrow \omega^2 \langle \hat{C}(t) \rangle = \hbar\omega \langle \hat{L}_3 - \hat{L}_1(t) \rangle$ oscillates at frequency $2\,\omega$

Pitaevskii / Rosch 1998



Saint-Jalm et al 2019

2D breathers



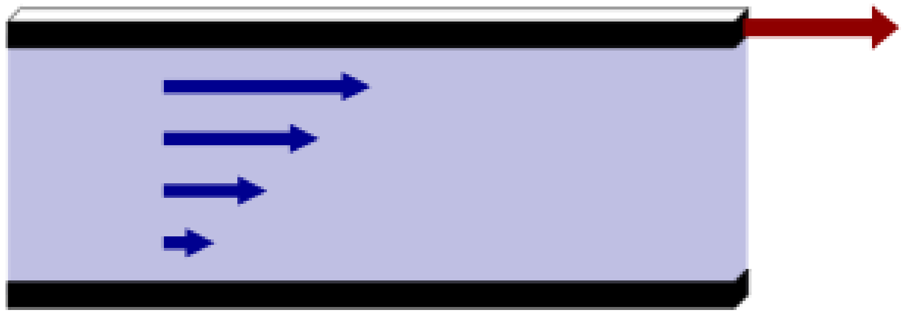
• from ideal to real fluids: effects of viscosity

$$\rho \frac{d\mathbf{v}}{dt} = -\mathrm{grad}\, p + \eta \nabla^2 \mathbf{v} + (\eta + \zeta/3)\mathrm{grad}\,\mathrm{div}\,\mathbf{v}$$ Navier-Stokes 1822 …

shear and bulk viscosity $\eta, \zeta \geq 0$ due to $dS/dt \geq 0$

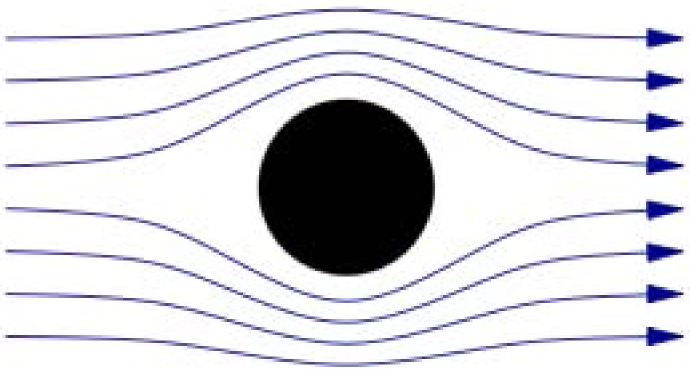
shear force per area

$\Pi_{xy} = \eta \cdot \partial_y v_x$ Newton 1695



friction force on a moving sphere

$\mathbf{F} = -6\pi R\, \eta \cdot \mathbf{v}$ Stokes 1851



stationary back flow current around a slowly moving obstacle

$$\delta\mathbf{j}(\mathbf{q}) = h(q)\,[(\hat{q}\cdot\mathbf{v})\hat{q} - \mathbf{v}] + \ldots$$

behavior at long distances $h(q) = \dfrac{h_{-2}}{q^2} + \dfrac{h_{-1}}{q} + h_0 + \ldots$ Zw. 1997

Stokes $h_{-2} = 6\pi n\, R$ and $h_0^{\text{Stokes}} = -(3/4)\, N_{\text{excluded}}$

dipolar back flow gives no drag force $\mathbf{F}^{\text{Stokes}} = -(h_{-2}/n)\,\eta\,\mathbf{v}$

superfluid flow has $h_{-2} = h_{-1} \equiv 0$ and $h_0^{\text{SF}} \simeq -2\, h_0^{\text{Stokes}} > 0$

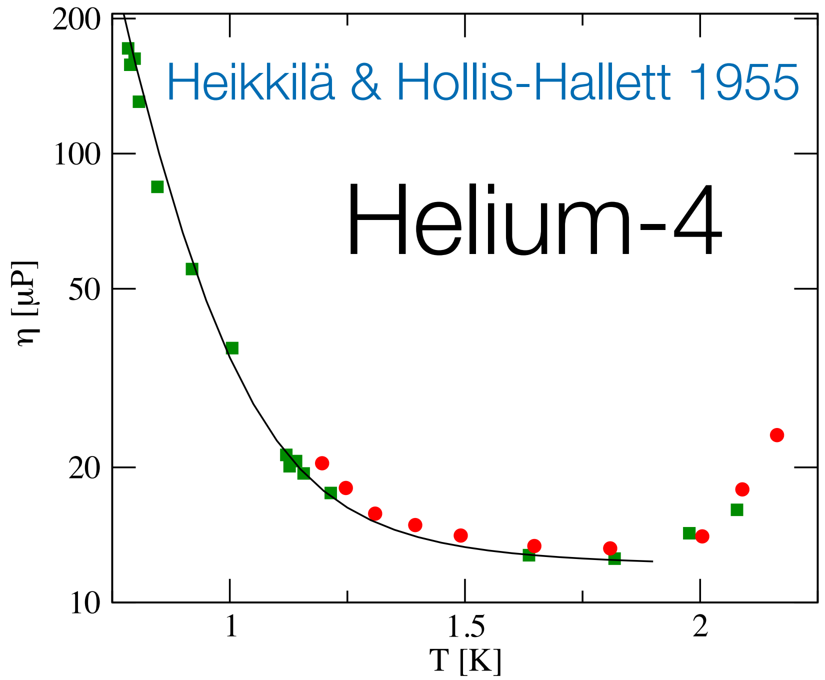
collisionless Fermi fluid $h_{-1} = 3\pi n\, \sigma_{\text{tr}}/4$ Landauer 1957

• a finite lower bound on viscosity ?

superfluids have $\eta^{(\mathrm{SF})} \equiv 0$ **but**

the normal component contributes

$\eta_n(T) \sim 1/T^5$ Landau / Khalatnikov 194[illegible]



kinetic theory of gases $\eta = mn\langle v\rangle\,\ell/3 \simeq \sqrt{mk_BT}/\sigma(T)$

grows as temperature increases and is independent of density !

Maxwell 1868

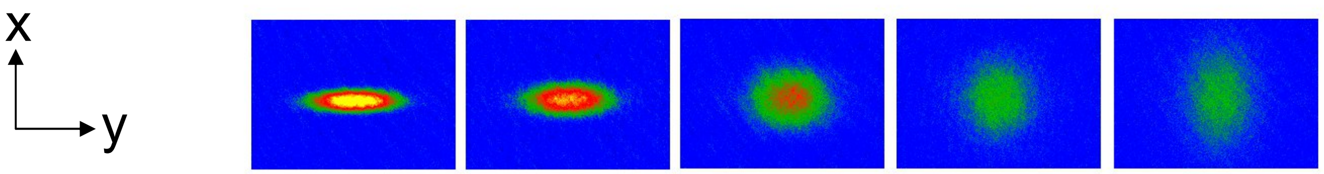
quantum limited viscosity

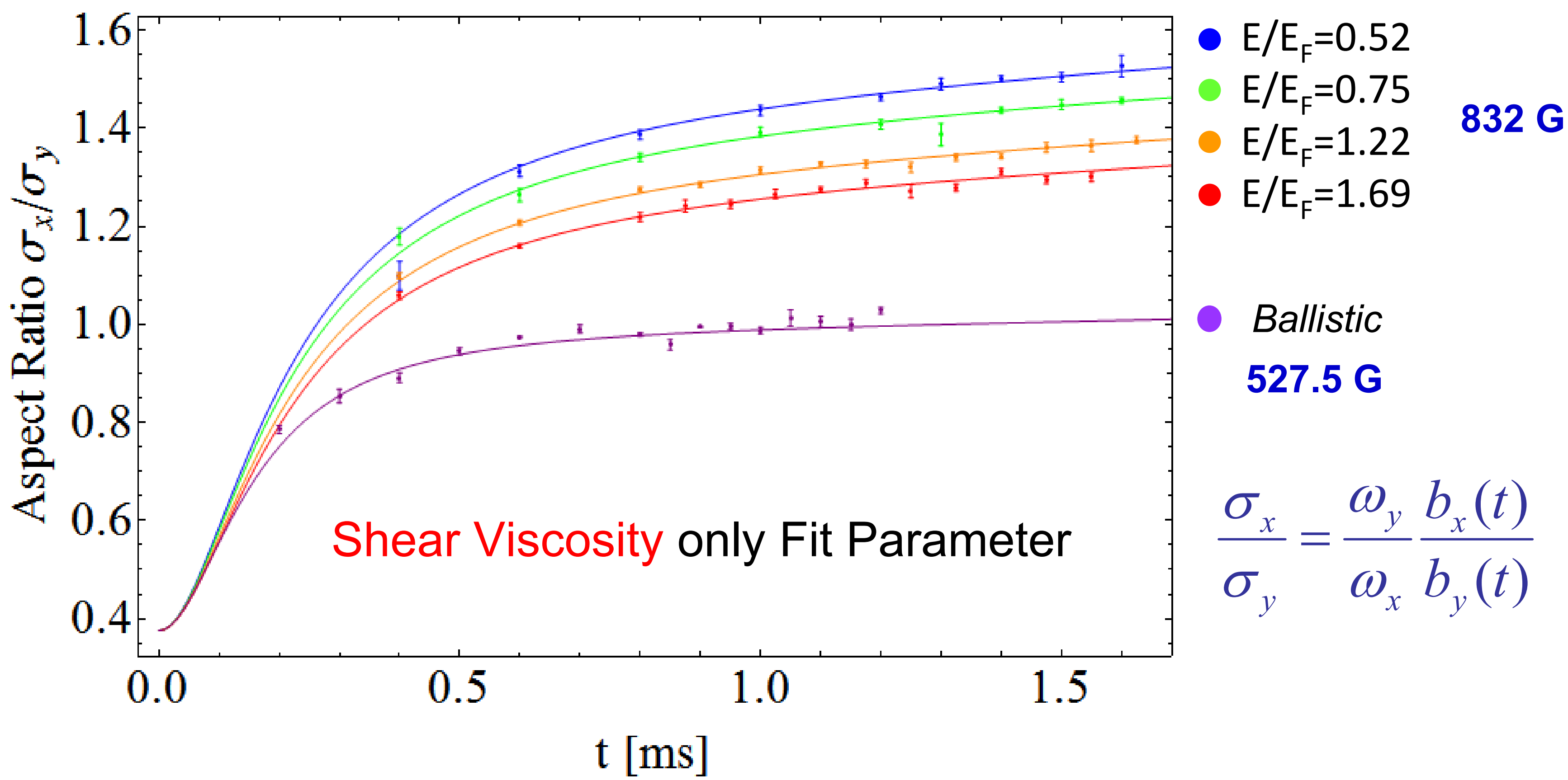
mean free path $\ell \gtrsim n^{-1/3}$ average velocity $\langle v\rangle \gtrsim (\hbar/m)\,n^{1/3}$

gives $\eta \geq \alpha_\eta \cdot \hbar n$ with $\alpha_\eta = \mathcal{O}(1)$ Shuryak 2005

inversion of the aspect ratio upon expansion of unitary fermions

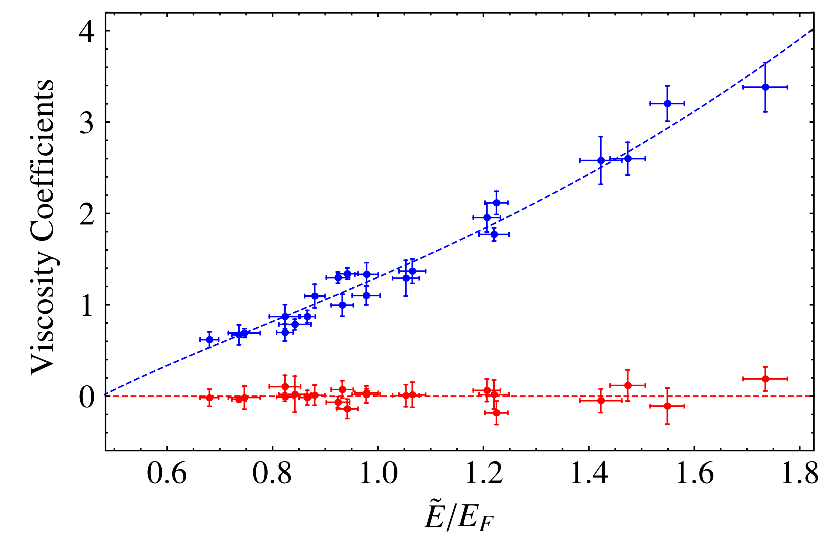
J. Thomas et al 2011 …





$$\frac{\sigma_x}{\sigma_y} = \frac{\omega_y}{\omega_x}\frac{b_x(t)}{b_y(t)}$$

extract η and ζ



elliptic flow of aspect ratio coexists with ballistic flow of $\langle \mathbf{r}^2 \rangle (t)$

• **vanishing bulk viscosity and a conjecture**

scale invariant fluids have vanishing bulk viscosity $\zeta(T) \equiv 0$

Werner/Castin 2006, Son 2007

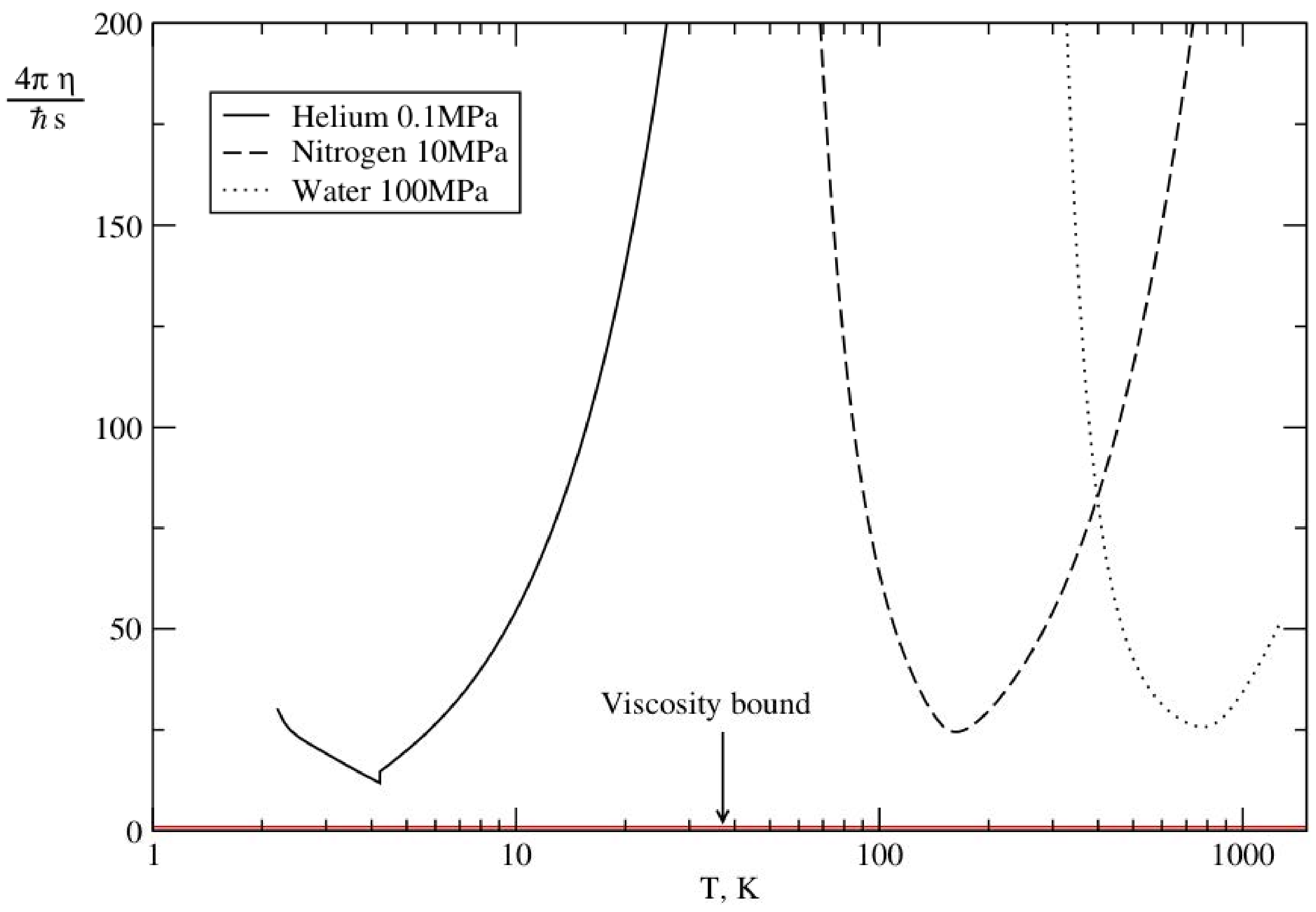
the shear viscosity to entropy density ratio is bounded below

$$\eta/s \geq \hbar/(4\pi k_B)$$

Kovtun/Son/Starinets 2005

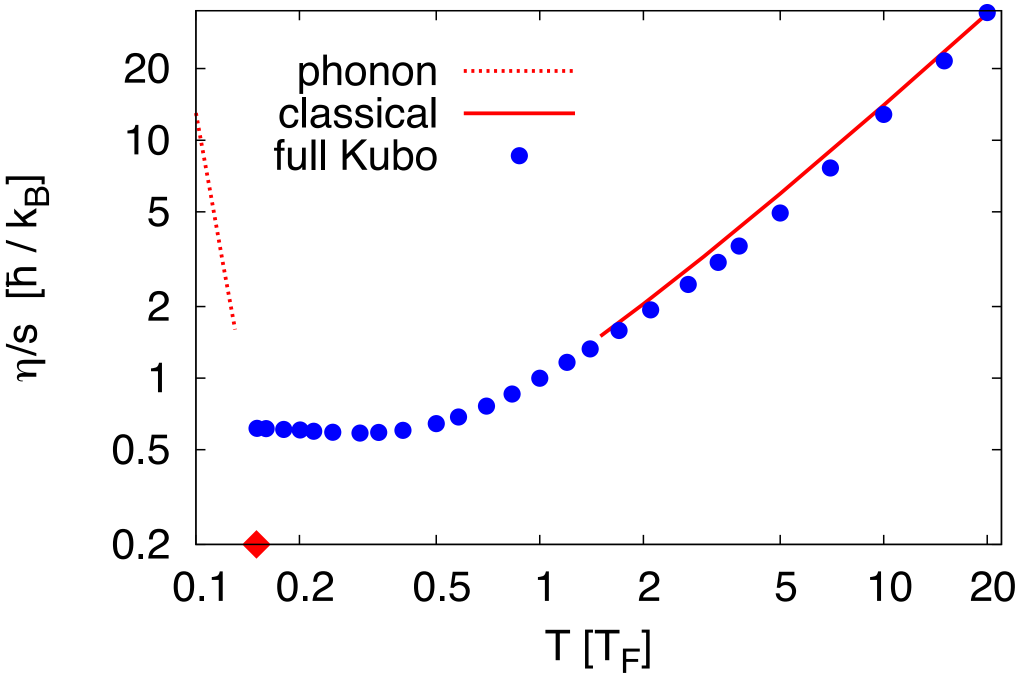
all known fluids so far

obey the bound !

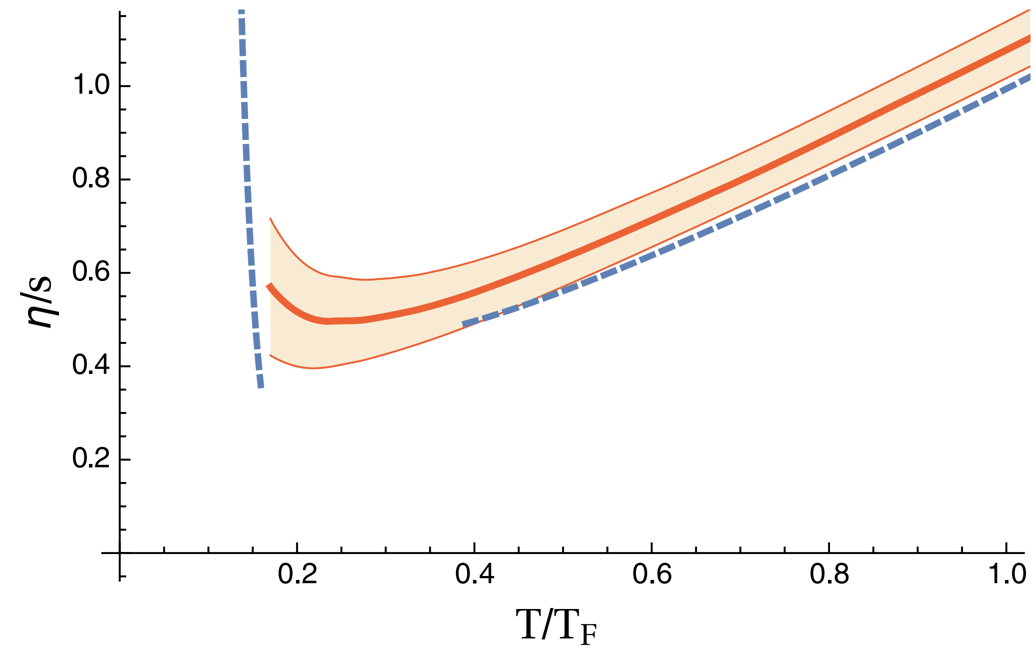


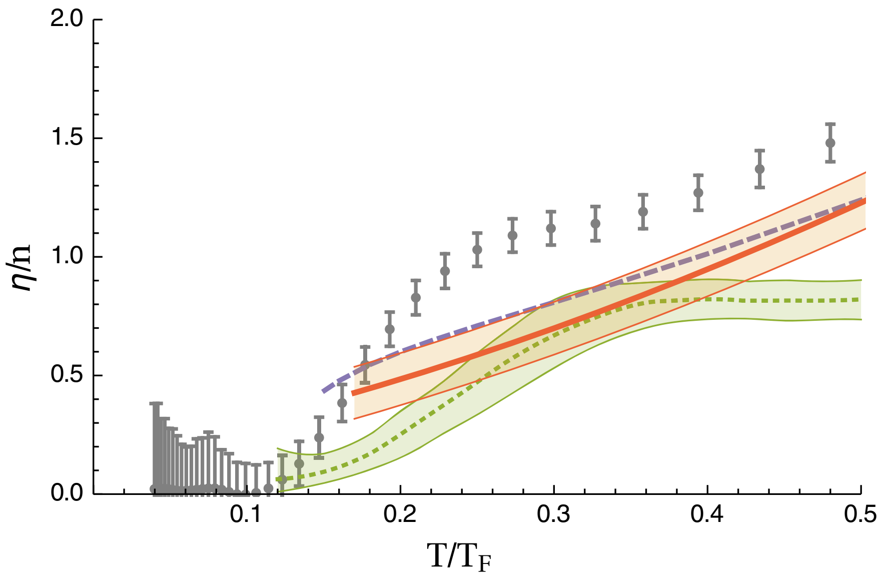
• a microscopic calculation of the shear viscosity

Kubo $$\mathrm{Re}\,\eta(\omega) = \frac{1}{\omega}\,\mathrm{Im}\int_0^\infty dt\, e^{i\omega t}\int_{\mathbf{x}}\langle \frac{i}{\hbar}[\hat{\Pi}_{xy}(\mathbf{x},t),\hat{\Pi}_{xy}(\mathbf{0},0)]\rangle$$



Enss / Haussmann / Zw. 2011

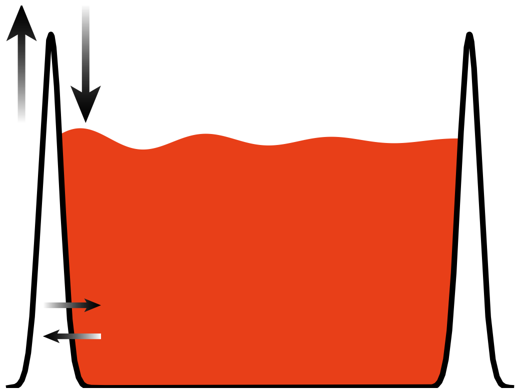


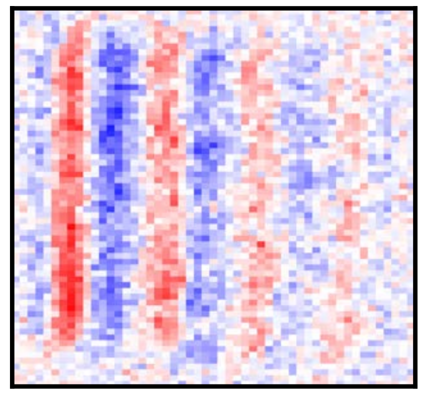


analysis of exp. data Bluhm et al 2017

the unitary Fermi gas has a minimum in $\eta/s \gtrsim 5\cdot\hbar/(4\pi k_B)$

• heat diffusion from damping of sound



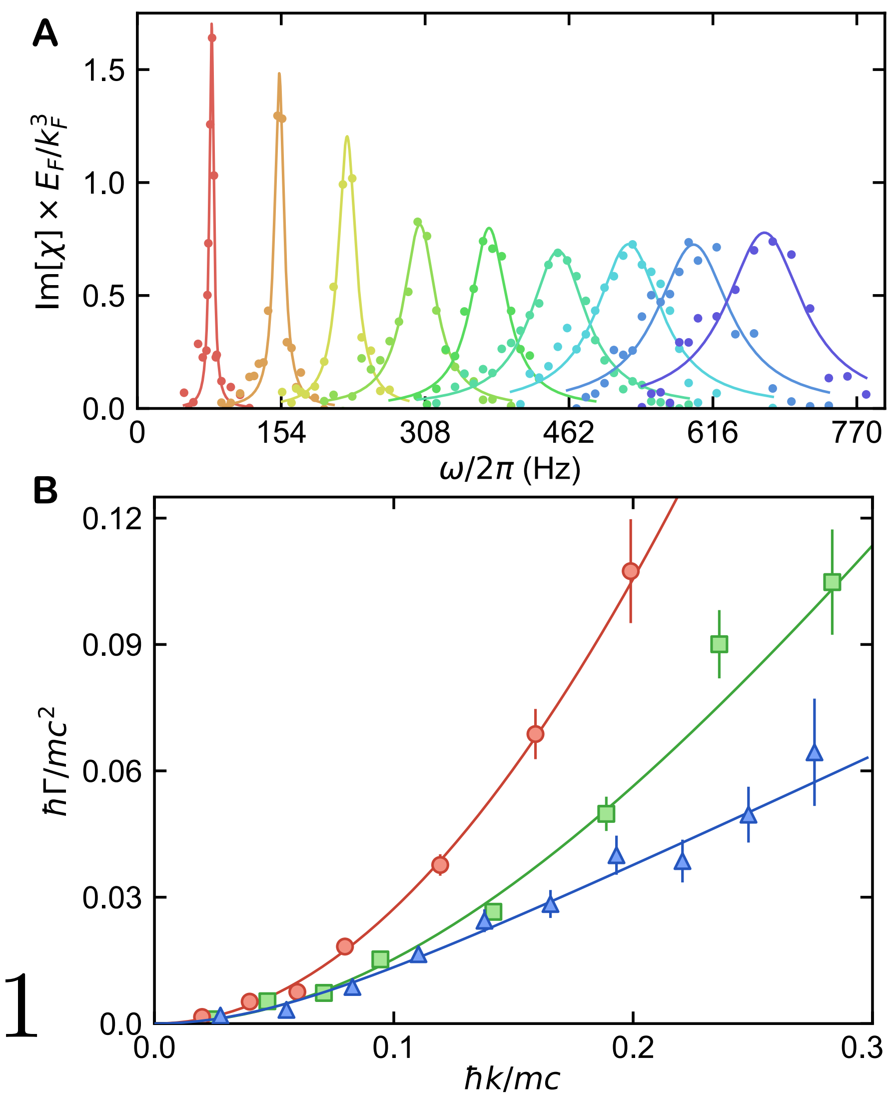


Patel et al 2020

sound waves of a unitary gas in a box

hydrodynamic damping

$$\omega_q = c_s q - iD_s q^2/2 + \ldots$$



sound diffusion constant

$$D_s = (4/3)\, D_\eta + \mathrm{LP} \cdot D_T \quad \text{with}$$

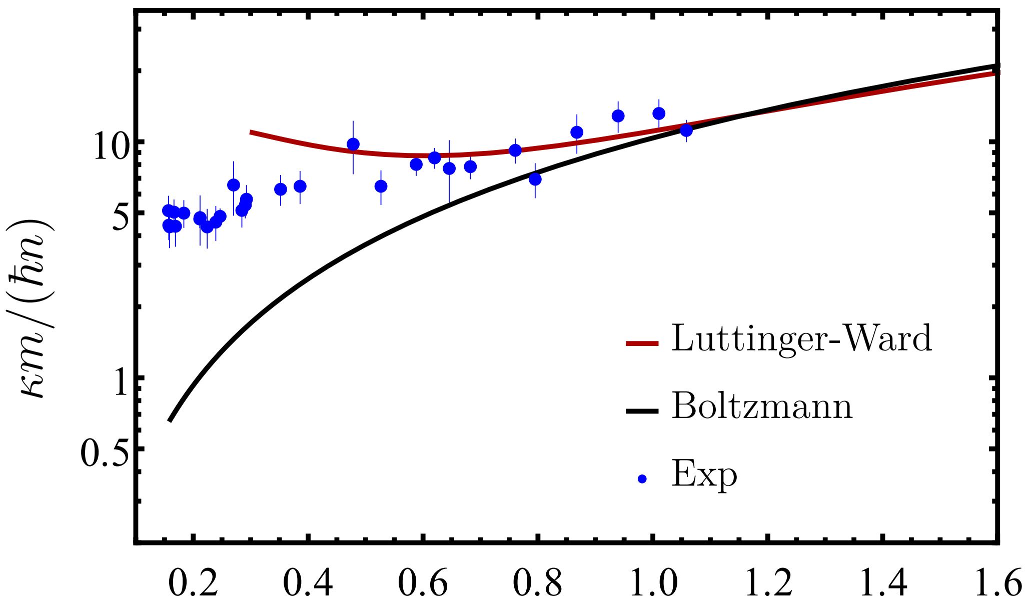
$$D_\eta = \eta/\rho \quad \text{and} \quad \mathrm{LP} = (c_p/c_V) - 1$$

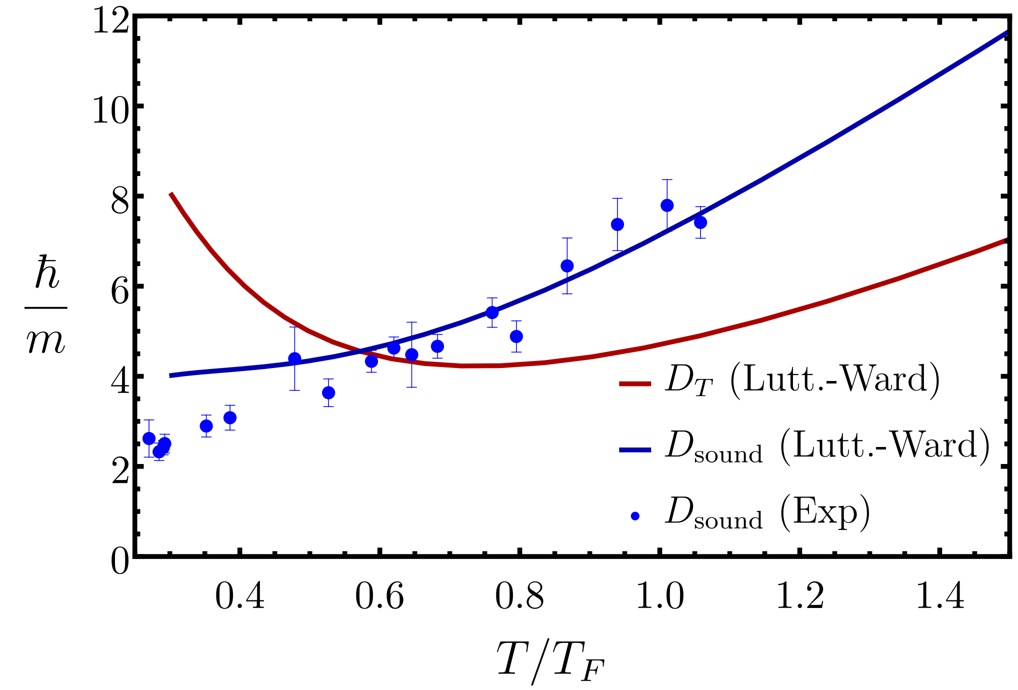
• quantum limit on heat diffusion

thermal conductivity $\quad \mathbf{j}^q = -\kappa \nabla T \quad$ Fourier 1822

Kubo formula requires $\quad [\hat{\mathbf{j}}^q(t), \hat{\mathbf{j}}^q(0)]$ where $\hat{\mathbf{j}}^q = \hat{\mathbf{j}}^E - \tilde{w}\,\hat{\mathbf{j}}$

thermal diffusion $\quad D_T = \kappa / c_p \quad$ diverges at SF transition





$(D_T)_{\text{min}} = 4.2\,\hbar/m \simeq 11\,(D_\eta)_{\text{min}}$ Frank/Zw./Enss 2020

- **broken scale invariance and bulk viscosity**

breaking of scale invariance gives rise to $\zeta \neq 0$

Schäfer 2013 Nishida 2018

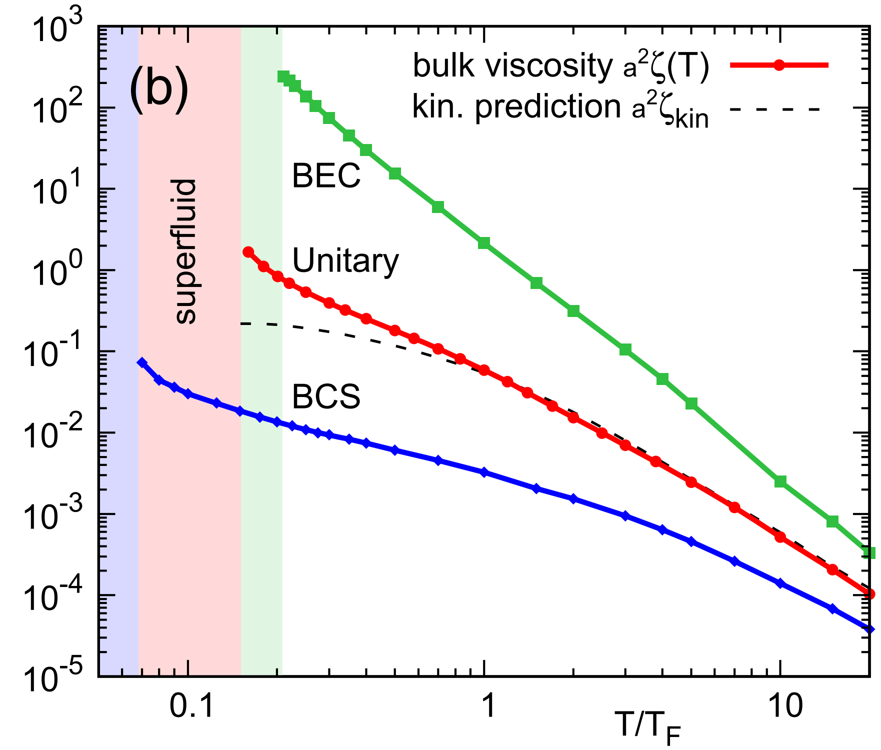
modulation $da^{-1}(t) = \alpha_0 \cos(\omega t)$ of the scattering length

yields $dE/dt \sim \alpha_0^2\, \omega\, \mathrm{Re}\, \zeta(\omega)$

$\zeta_{\mathrm{dc}}(T)$ grows near the SF transition

anomaly in 2D Bose gases gives finite

$$\zeta_{\mathrm{2D}}(\omega = 0) = \frac{\hbar n \cdot 2\pi\, n\lambda_T^2}{\left[\ln^2(k_B T/2\varepsilon_b) + \pi^2\right]^2}$$



Enss 2019 unitary fermions

• Conclusion

- Ultracold gases provide a number of concrete realisations for scale invariant many-body problems in 1D, 2D and 3D. The additional symmetries give rise to a universal equation of state and to unique features in the dynamics.

- Transport coefficients in scale invariant gases approach quantum limited values, associated with relaxation times of order $\hbar/k_B T$. Deviations from scale invariance lead to a non-vanishing bulk viscosity which appears e.g. in the response to a time dependent change of the scattering length.

Au revoir et Merci beaucoup pour votre attention