Vacuum fluctuations and radiation reaction : identification of their respective contributions

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Résumé. — Il semble généralement admis qu'il existe, en théorie quantique du rayonnement, une indétermination dans la séparation des effets respectifs des fluctuations du vide et de la réaction de rayonnement. Nous montrons ici que cette indétermination est levée si l'on impose aux vitesses de variation correspondantes d'être hermitiques (condition nécessaire pour qu'elles soient interprétables physiquement). Cette procédure est généralisée au cas d'un petit système S interagissant avec un grand réservoir ℜ, et permet de séparer deux types de processus physiques, ceux où ℜ fluctue et polarise S (effets des fluctuations du réservoir), ceux où c'est S qui polarise ℜ (effets de la réaction de ℜ sur S). Nous appliquons cette procédure au cas d'un électron atomique interagissant avec le champ de rayonnement et identifions ainsi les contributions des fluctuations du vide et de la réaction de rayonnement aux corrections radiatives et à l'émission spontanée. L'analyse des résultats obtenus nous permet de préciser les images physiques qui doivent être associées aux divers processus radiatifs.

Abstract. — It is generally considered that there exists in quantum radiation theory an indetermination in the separation of the respective effects of vacuum fluctuations and radiation reaction. We show in this paper that such an indetermination can be removed by imposing to the corresponding rates of variation to be Hermitian (this is necessary if we want them to have a physical meaning). Such a procedure is generalized to the case of a small system S interacting with a large reservoir ℜ and allows the separation of two types of physical processes, those where ℜ fluctuates and polarizes S (effects of reservoir fluctuations), those where it is S which polarizes ℜ (effects of self reaction). We apply this procedure to an atomic electron interacting with the radiation field and we then identify the contribution of vacuum fluctuations and self reaction to radiative corrections and spontaneous emission of radiation. The analysis of the results obtained in this way allows us to specify the physical pictures which must be associated with the various radiative processes.

1. **Introduction.** — Understanding the physical mechanisms responsible for spontaneous emission of radiation by an excited atom, or for radiative corrections such as radiative line shifts, electron's self energy or magnetic moment... is a very stimulating problem which has received a lot of attention [1, 2].

The quantitative results for these corrections are of course well established. The physical interpretations remain however more controversial. Two extreme points of view have been investigated. In the first one, the interaction of the electron with the quantum fluctuations of the vacuum field, the so-called « vacuum fluctuations », is considered as playing the central role. One tries to interpret spontaneous emission as an emission « triggered » by vacuum fluctuations. The most famous example of such an approach is the interpretation of the Lamb shift as being due to

the averaging of the Coulomb potential of the nucleus by the electron vibrating in vacuum fluctuations [3]. One must not forget however that such a picture leads to the wrong sign for the electron's spin anomaly $g - 2$: the vibration of the electron's spin in vacuum fluctuations does not increase the effective magnetic moment but reduces it [3, 4]. In the second point of view, the basic physical mechanism is identified as the interaction of the electron with its own field, the so called « radiation reaction » although it would be proper to call it the electromagnetic self interaction since it includes the interaction of the electron with its Coulomb field as well as with its radiation field [5-8]. We will use in the following the shorter denomination « self reaction » for this process. In such an approach, one tries to interpret Q.E.D. radiative corrections along the same lines as the radiative damping and

the radiative shift of an oscillating classical dipole moment. We should note however that the vacuum field cannot be completely forgotten in the interpretation of finer details of spontaneous emission, such as fluorescence spectrum or intensity correlations, which are related to higher order correlation functions [9, 10].

Actually, it is now generally accepted that vacuum fluctuations and self reaction are « two sides of the same quantum mechanical coin » [11], and that their respective contributions to each physical process cannot be unambiguously determined [11-14]. Such an opinion is based on the following analysis, carried out in the Heisenberg picture which provides a very convenient theoretical framework since it leads, for the relevant dynamical variables, to equations of motion very similar to the corresponding classical ones. The calculations [11-14] can be summarized by the general scheme of figure 1.

Heisenberg's equations of motion for field and atomic variables are derived from the Hamiltonian of the combined atom + field system. The equation for the field looks like the equation of motion of an harmonic oscillator driven by an atomic source term and is readily integrated. This leads to an expression for the total field E which is a sum of two terms :

$$E = E_f + E_s .$$
(1.1)

The « free field » E_f corresponds to the solution of the homogeneous field equation (without atomic source term), and coincides with the « vacuum field » when no photons are initially present. The « source field » E_s is the field generated by the atomic source (solution of the inhomogeneous field equation). Consider now the atomic equation. The rate of variation, $dG(t)/dt$, of a given atomic observable $G(t)$ appears to be proportional to the product of atomic and field operators, $N(t)$ and $E(t)$, taken at the same time :

$$\frac{dG(t)}{dt} \sim N(t) E(t) .$$
(1.2)

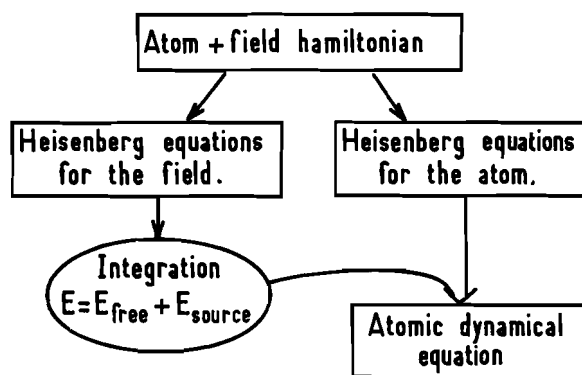The final step of the calculation consists in inserting in (1.2) the solution (1.1) obtained for $E(t)$, which

leads to a dynamical equation for the atomic system (Fig. 1). The contributions of E_f and E_s to dG/dt can be interpreted as rates of variation :

$$\left(\frac{dG}{dt}\right)_{vf} \sim N(t) E_f(t)$$
(1.3a)

$$\left(\frac{dG}{dt}\right)_{sr} \sim N(t) E_s(t)$$
(1.3b)

respectively due to vacuum fluctuations and self reaction. This interpretation directly follows from the physical origin of E_f and E_s. The ambiguity mentioned above for this separation comes from the fact that the two atomic and field operators $N(t)$ and $E(t)$ appearing in (1.2) commute [they commute at the initial time $t = t_0$, when they act in different spaces, and the Hamiltonian evolution between t_0 and t preserves this commutation]. They can therefore be taken in any order, $N(t) E(t)$ as in (1.2), or $E(t) N(t)$. However, $E_f(t)$ and $E_s(t)$ do not commute separately with $N(t)$, as their sum does. Consequently, $N(t) E_f(t)$ and $E_f(t) N(t)$ generally differ. The two rates of variation (1.3a) and (1.3b) therefore depend on the initial order between the two commuting operators $N(t)$ and $E(t)$, the total rate (1.2) being of course independent of this order. In particular, if the normal order has been chosen in (1.2) [with all field annihilation operators at right, all field creation operators at left], the contribution of vacuum fluctuations vanishes when the average is taken over the vacuum state of the field, and all radiative corrections appear to come from self reaction. Different orders taken in (1.2) would lead to different conclusions. Thus, it seems that the relative contributions of vacuum fluctuations and self reaction cannot be unambiguously identified.

MOTIVATIONS OF THIS PAPER. — In this paper, we would like to present some arguments supporting the choice of a particular order in (1.2) leading, in our opinion, to a physically well defined separation between the contributions of vacuum fluctuations and self reaction. We don't question of course the mathematical equivalence of all possible initial orders in (1.2). Our argument rather concerns the physical interpretation of the two rates of variation appearing when (1.1) is inserted in (1.2). If G is an atomic observable (Hermitian operator), the two rates of variation contributing to $\frac{d}{dt} G(t)$, which is also Hermitian, must be separately Hermitian, if we want them to have a physical meaning. Furthermore, the field and atomic operators appearing in the different rates of variation must also be Hermitian if we want to be able to analyse these rates in terms of well defined physical quantities. We show in this paper that these hermiticity requirements restrict the possible initial orders in (1.2) to only one, the completely symmetrical order.

Fig. 1. — Principle of the derivation of the atomic dynamical equation.

A second motivation of this paper is to point out that, with such a symmetrical order, a clear connection can be made with a statistical mechanics point of view which appears to be in complete agreement with the usual physical pictures associated with vacuum fluctuations and self reaction. For example, the radiative corrections can be expressed as products of correlation functions by linear susceptibilities. For the vacuum fluctuations part of these corrections, one gets the correlation function of the field multiplied by the linear susceptibility of the atom, which supports the picture of a fluctuating vacuum field polarizing the atomic system and interacting with this induced polarization, whereas for the self reaction part, the reverse result is obtained : product of the correlation function of the atomic system by the linear « susceptibility » of the field which corresponds to the picture of a fluctuating dipole moment « polarizing » the field, i.e. producing a field, and interacting with this field.

ORGANIZATION OF THE PAPER. — In section 2 we introduce our notations and the basic concepts (vacuum field, source field, radiation reaction...) by applying the general theoretical scheme of figure 1 to the derivation of the quantum generalization of the Abraham-Lorentz equation [17] describing the dynamics of an atomic electron interacting with a static potential and with the quantized radiation field. We discuss the physical content of this equation and the difficulties associated with the quantum nature of field variables. We explain also why it is necessary to extend the calculations of section 2 (dealing with the position $\mathbf{r}$ and the momentum $\mathbf{p}$ of the electron) to more general atomic observables G.

The calculation of dG/dt, which is presented in section 3, raises the problem of the order between commuting observables, mentioned above in connection with equation (1.2) (such a difficulty does not appear for $\mathbf{r}$ and $\mathbf{p}$). We show how it is possible, by the physical considerations mentioned above, to single out the completely symmetrical order in (1.2). We then extend in section 4 the discussion to the more general case of a « small system » S (playing the role of the atomic system) interacting with a « large reservoir » $\mathcal{R}$ (playing the role of the electromagnetic field with its infinite number of degrees of freedom). The advantage of such a generalization is to provide a deeper insight in the problem. We point out in particular that the expressions giving $\left\langle \left(\dfrac{dG}{dt} \right)_{vf} \right\rangle$ and $\left\langle \left(\dfrac{dG}{dt} \right)_{sr} \right\rangle$, averaged in the vacuum state of the field and calculated to the first order in the fine structure constant α, can be expressed in terms of simple statistical functions of the two interacting systems (correlation functions and linear susceptibilities). We discuss the mathematical structure of these expressions and their physical content.

Finally, the general results of sections 3 and 4 are applied in section 5 to the physical discussion of the relative contributions of vacuum fluctuations and self reaction to the dynamics of an atomic electron. Two types of effects are considered : the shift of atomic energy levels, described by the Hamiltonian part of $\left\langle \left(\dfrac{dG}{dt} \right)_{vf} \right\rangle$ and $\left\langle \left(\dfrac{dG}{dt} \right)_{sr} \right\rangle$, and the dissipative effects associated with the exchange of energy between the electron and the radiation field.

2. The quantum form of the Abraham-Lorentz equation.

— A few basic concepts are introduced in this section, by considering a very simple system formed by an electron bound near the origin by an external potential and interacting with the electromagnetic field.

We first introduce the Hamiltonian of the combined system « bound electron + electromagnetic field » (§ 2.1). We then establish, in the Heisenberg representation, the quantum dynamical equation for the electron (§ 2.2). This equation appears to be very similar · to the corresponding classical one, known as the Abraham-Lorentz equation. This close analogy is however misleading and we will try to explain the difficulties hidden in the quantum equation (§ 2.3).

2.1 BASIC HAMILTONIAN IN COULOMB GAUGE. —

2.1.1 *Field variables.* — The electric field is divided into two parts : the longitudinal field $\mathbf{E}_{\parallel}$ and the transverse field $\mathbf{E}_{\perp}$. The longitudinal field at point $\mathbf{R}$ is the instantaneous Coulomb field created by the electron at this point. It is expressed as a function of the electron position operator $\mathbf{r}$.

$$\mathbf{E}_{\parallel}(\mathbf{R}) = - \mathbf{V}_{\mathbf{R}} \frac{e}{4 \pi \varepsilon_0 | \mathbf{R} - \mathbf{r} |}. \qquad (2.1)$$

The transverse field $\mathbf{E}_{\perp}(\mathbf{R})$, the vector potential $\mathbf{A}(\mathbf{R})$ and the magnetic field $\mathbf{B}(\mathbf{R})$ are expanded in a set of transverse plane waves, normalized in a cube of volume L^3 :

$$\mathbf{E}_{\perp}(\mathbf{R}) = \sum_{k\varepsilon} (\mathcal{E}_k \, \varepsilon e^{i k \mathbf{R}}) \, a_{k\varepsilon} + hc \qquad (2.2a)$$

$$\mathbf{A}(\mathbf{R}) = \sum_{k\varepsilon} (\mathcal{A}_k \, \varepsilon e^{i k \mathbf{R}}) \, a_{k\varepsilon} + hc \qquad (2.2b)$$

$$\mathbf{B}(\mathbf{R}) = \sum_{k\varepsilon} (\mathcal{B}_k \, \kappa \times \varepsilon e^{i k \mathbf{R}}) \, a_{k\varepsilon} + hc \qquad (2.2c)$$

with :

$$\mathcal{E}_k = i \sqrt{\frac{\hbar \omega}{2 \, \varepsilon_0 \, L^3}}, \quad \mathcal{A}_k = \sqrt{\frac{\hbar}{2 \, \varepsilon_0 \, L^3 \, \omega}}, \quad \mathcal{B}_k = i \frac{\omega \mathcal{A}_k}{c}, \qquad (2.3)$$

$$\kappa = \frac{\mathbf{k}}{k} \qquad (2.4)$$

$a_{\mathbf{k}\varepsilon}$ and $a_{\mathbf{k}\varepsilon}^+$ are the annihilation and creation operators for a photon with wave vector $\mathbf{k}$ and polarization ε. The summation concerns all the wave vectors $\mathbf{k}$ with components multiple of $2\pi/L$ and, for a given $\mathbf{k}$, two transverse orthogonal polarizations ε_1 and ε_2.

In classical theory, expansions similar to (2.2) can be written, the operators $a_{\mathbf{k}\varepsilon}$ and $a_{\mathbf{k}\varepsilon}^+$ being replaced by c-numbers $\alpha_{\mathbf{k}\varepsilon}(t)$ and $\alpha_{\mathbf{k}\varepsilon}^*(t)$ which are actually « normal » variables for the field.

In order to calculate the energy of the Coulomb field of the particle, it is also convenient to take the Fourier transform of the longitudinal field (2.1) (for a given value of $\mathbf{r}$) :

$$\mathbf{E}_{\parallel}(\mathbf{R}) = \sum_{\mathbf{k}} - i\,\frac{e}{2\,\varepsilon_0\,L^3\,k}\,\mathbf{\kappa}\,e^{i\mathbf{k}\cdot(\mathbf{R}-\mathbf{r})} + hc\,. \quad (2.5)$$

2.1.2 Electron variables.

— The electron motion is described by the position operator $\mathbf{r}$ and the conjugate momentum $\mathbf{p}$:

$$\mathbf{p} = \frac{\hbar}{i}\,\mathbf{\nabla}_{\mathbf{r}}\,. \quad (2.6)$$

The velocity operator, $\mathbf{v}$, is given by :

$$m\mathbf{v} = \mathbf{p} - e\mathbf{A}(\mathbf{r}) \quad (2.7)$$

where m is the electron mass. Note that $\mathbf{v}$ is not an electronic operator since it includes field variables through $\mathbf{A}(\mathbf{r})$. The electron is bound near the origin by an external static potential $V_0(\mathbf{R})$. If spin is taken into account, the electron variables are supplemented by the spin operator $\mathbf{S}$. Magnetic and spin effects will be briefly discussed in § 5.2.5. They are neglected elsewhere.

2.1.3 The Hamiltonian.

— In the non relativistic approximation, the Hamiltonian is the sum of five terms : the rest mass energy of the electron, its kinetic energy, its potential energy in $V_0(\mathbf{R})$, the energy of the longitudinal field and the energy of the transverse fields :

$$H = mc^2 + \frac{1}{2\,m}\,(\mathbf{p} - e\mathbf{A}(\mathbf{r}))^2 +$$

$$+ V_0(\mathbf{r}) + \frac{\varepsilon_0}{2}\int d^3\,R\,\mathbf{E}_{\parallel}^2(\mathbf{R})$$

$$+ \sum_{\mathbf{k}\varepsilon}\frac{\hbar\omega}{2}\,(a_{\mathbf{k}\varepsilon}^+\,a_{\mathbf{k}\varepsilon} + a_{\mathbf{k}\varepsilon}\,a_{\mathbf{k}\varepsilon}^+)\,. \quad (2.8)$$

The energy of the longitudinal field appears to be a constant, representing the energy of the electrostatic field associated with the charge. This constant can be written as

$$\frac{\varepsilon_0}{2}\int d^3 R\,\mathbf{E}_{\parallel}^2(\mathbf{R}) = \delta m_1\,c^2 \quad (2.9)$$

δm_1 can be considered as a correction to the mechanical rest mass m of the electron. The same correction appears in classical theory.

2.1.4 Introduction of a cut-off.

— It is well known that divergences appear in the computation of various physical quantities (such as energy, momentum...) associated with a charged point particle interacting with the electromagnetic field. These divergences are due to the contribution of the modes with large wave vectors. In order to deal with finite expressions, we will consider only the coupling of the electron with modes $\mathbf{k}$ such that

$$|\,\mathbf{k}\,| < k_M\,. \quad (2.10)$$

This cut-off k_M is chosen not too large so that the non relativistic approximation is correct for all the modes which are taken into account ($\hbar\omega_M \ll mc^2$ with $\omega_M = ck_M$). On the other hand, ω_M must be large compared to the characteristic resonance frequencies ω_0 of the bound electron. This gives two bounds for k_M :

$$\frac{\omega_0}{c} \ll k_M \ll \frac{mc}{\hbar}\,. \quad (2.11)$$

It is well known that theories using such a cut-off are no longer relativistic invariant [15]. The modes selected by condition (2.10) are not the same in two different reference frames, because of the Doppler effect. It is possible to restore relativistic invariance, by using some more sophisticated cut-off procedures [16]. However, we are not concerned here with the relativistic aspects of radiative problems and we will ignore these difficulties. To summarize, all the sums over $\mathbf{k}$ appearing here after must be understood as limited by condition (2.10). The same restriction also applies to the expansion (2.5) of the longitudinal field. The energy of the longitudinal field is then finite and equal to

$$\delta m_1\,c^2 = \sum_{\mathbf{k}}\frac{e^2}{2\,\varepsilon_0\,L^3\,k^2} = \frac{e^2\,k_M}{4\,\pi^2\,\varepsilon_0} \quad (2.12)$$

which can be written as $\dfrac{\alpha}{\pi}\,\hbar\omega_M$, where α is the fine structure constant.

2.1.5 Electric dipole approximation.

— We also suppose in this paper that the binding potential localizes the electron in a volume centred on the origin, with a linear dimension a much smaller than the wave-length of the modes interacting with the particle. (The cut-off k_M introduced above is supposed to satisfy $k_M\,a \ll 1$.) Such an assumption which is justified for an atomic electron, allows us to neglect the spatial variation of the fields interacting with the electron. We will then replace the fields at the electron position $\mathbf{E}(\mathbf{r})$, $\mathbf{A}(\mathbf{r})$ by the fields at the origin $\mathbf{E}(0)$, $\mathbf{A}(0)$.

The electric dipole approximation is not essential for the derivation of the results presented in this paper. But the calculations are much simpler and the physical conclusions remain unchanged ([1]).

([1]) Corrections to the electric dipole approximation are of higher order in $1/c$. They have to be considered when relativistic corrections are included in the Hamiltonian (see for example [4]).

To summarize the previous discussion, we will use hereafter the following Hamiltonian :

$$H_{ED} = (m + \delta m_1) c^2 + \frac{1}{2\,m}(\mathbf{p} - e\mathbf{A}(0))^2 + V_0(\mathbf{r}) + \sum_{\substack{k\epsilon \\ |k| < k_M}} \frac{\hbar\omega}{2}\,(a^+_{k\epsilon}\,a_{k\epsilon} + a_{k\epsilon}\,a^+_{k\epsilon}) \qquad (2.13)$$

with

$$\mathbf{A}(0) = \sum_{\substack{k\epsilon \\ |k| < k_M}} \mathcal{A}_k\,\boldsymbol{\epsilon}\,a_{k\epsilon} + hc. \qquad (2.14)$$

2.2 DYNAMICS OF THE ELECTRON INTERACTING WITH THE ELECTROMAGNETIC FIELD. — 2.2.1 *Principle of the calculation.* — The rate of variation of electron and field variables can be determined from the Hamiltonian (2.13). The corresponding two sets of equations are of course coupled; the field evolution depends on the charge motion and, conversely, the electron experiences a force due to the field.

The derivation of a dynamical equation for the electron from these two sets of coupled equations is well known [8, 13, 14] and follows the general scheme of figure 1. One first integrates the field equations in presence of the particle. The solution obtained for the field is then inserted in the electron equation. This leads to a quantum dynamical equation describing the motion of the electron interacting with the free field as well as with its own field.

2.2.2 *The electromagnetic field in presence of the particle.* — Since all field operators are expressed in terms of $a_{k\epsilon}$ and $a^+_{k\epsilon}$, we start with the Heisenberg equation for $a_{k\epsilon}(t)$:

$$\dot{a}_{k\epsilon}(t) = \frac{i}{\hbar}[H(t), a_{k\epsilon}(t)] = -i\omega a_{k\epsilon}(t) + \frac{ie}{m\hbar}\mathcal{A}_k\,\boldsymbol{\epsilon}^*.\boldsymbol{\pi}(t) \qquad (2.15)$$

where

$$\boldsymbol{\pi}(t) = m\mathbf{v}(t) = \mathbf{p}(t) - e\mathbf{A}(0, t). \qquad (2.16)$$

Equation (2.15) is then formally integrated and gives :

$$a_{k\epsilon}(t) = a_{k\epsilon}(t_0)\,e^{-i\omega(t-t_0)} + \frac{ie}{m\hbar}\mathcal{A}_k \int_{t_0}^t dt'\,e^{-i\omega(t-t')}\,\boldsymbol{\epsilon}^*.\boldsymbol{\pi}(t'). \qquad (2.17)$$

The evolution of $a_{k\epsilon}(t)$ appears to be the superposition of a free evolution [first term of (2.17)] and a « forced » evolution driven by the motion of the charge [second term of (2.17)]. We finally insert (2.17) in the expansions (2.2) of the transverse field. The contributions of the two terms of (2.17) correspond respectively to the free fields ($\mathbf{A}_f$, $\mathbf{E}_{\perp f}$) and to the source fields ($\mathbf{A}_s$, $\mathbf{E}_{\perp s}$). Actually, we need only for the following to know the fields for $\mathbf{R} = 0$ (because of the electric dipole approximation). From (2.2) and (2.17), one easily derives (see appendix A for the details of the calculation) :

$$\mathbf{A}(0, t) = \mathbf{A}_f(0, t) + \mathbf{A}_s(0, t) \qquad (2.18)$$

with

$$\mathbf{A}_f(0, t) = \sum_{k\epsilon}(\mathcal{A}_k\,\boldsymbol{\epsilon}\,e^{-i\omega(t-t_0)})\,a_{k\epsilon}(t_0) + hc \qquad (2.19a)$$

$$\mathbf{A}_s(0, t) = \frac{4}{3}\frac{\delta m_1}{me}\boldsymbol{\pi}(t) - \frac{2}{3}\frac{e}{4\pi\varepsilon_0\,mc^3}\dot{\boldsymbol{\pi}}(t) \qquad (2.19b)$$

similarly

$$\mathbf{E}_\perp(0, t) = \mathbf{E}_{\perp f}(0, t) + \mathbf{E}_{\perp s}(0, t) \qquad (2.20)$$

with

$$\mathbf{E}_{\perp f}(0, t) = \sum_{k\epsilon}(\mathcal{E}_k\,\boldsymbol{\epsilon}\,e^{-i\omega(t-t_0)})\,a_{k\epsilon}(t_0) + hc \qquad (2.21a)$$

$$\mathbf{E}_{\perp s}(0, t) = -\frac{4}{3}\frac{\delta m_1}{me}\dot{\boldsymbol{\pi}}(t) + \frac{2}{3}\frac{e}{4\pi\varepsilon_0\,mc^3}\ddot{\boldsymbol{\pi}}(t). \qquad (2.21b)$$

2.2.3 *The quantum Abraham-Lorentz equation.* — The Heisenberg equations for the electron operators r and π are

$$m\dot{\mathbf{r}}(t) = \frac{im}{\hbar}[H, \mathbf{r}] = \pi(t) \tag{2.22}$$

$$\dot{\pi}(t) = \frac{i}{\hbar}[H, \pi] = -\nabla V_0(\mathbf{r}) + e\mathbf{E}_\perp(0, t) + \frac{e}{2m}(\pi \times \mathbf{B}(0, t) - \mathbf{B}(0, t) \times \pi). \tag{2.23}$$

The last term of the right member of equation (2.23) is smaller than the second one by a factor v/c [see Eq. (2.3)]. It will be neglected hereafter. On the other hand, we notice that $\mathbf{E}_\perp(0, t)$ is not multiplied by any electronic operator so that the problem of order raised in the introduction does not appear here. Replacing in (2.23) the total transverse electric field by the sum (2.20) of the free field and the source field and using (2.22) to eliminate π, one gets :

$$m\ddot{\mathbf{r}}(t) = -\nabla V_0(\mathbf{r}) + e\mathbf{E}_{\perp s}(0, t) + e\mathbf{E}_{\perp f}(0, t)$$

$$= -\nabla V_0(\mathbf{r}) - \frac{4}{3}\delta m_1 \ddot{\mathbf{r}}(t) + \frac{2}{3}\frac{e^2}{4\pi\varepsilon_0 c^3}\dddot{\mathbf{r}}(t) + e\mathbf{E}_{\perp f}(0, t). \tag{2.24}$$

This equation is very similar to the classical Abraham-Lorentz equation [17]. This is not surprising since the classical Hamiltonian is similar to (2.13). The general scheme of figure 1 is valid for both quantum and classical theories, and the Hamilton-Jacobi equations have the same structure as the quantum Heisenberg ones. Since there is no problem of order, the physical interpretation of this equation is clear. Apart from the external potential $V_0(\mathbf{r})$, two fields act on the electron : its own field and the free field. The coupling of the electron with its own field is described by two terms : the first one, proportional to $\ddot{\mathbf{r}}$, correspond to a mass renormalization from m to $m + \frac{4}{3}\delta m_1$ (^{2}). The second one proportional to $\dddot{\mathbf{r}}$, is the quantum analogue of the force which produces the radiative damping of the classical particle. The last term of (2.24) describes the coupling of the electron with the free field, i.e. the field which would exist if the particle was not there. This free field may include an incident radiation field. Classically, the description of the electron free motion is obtained by taking $\mathbf{E}_{\perp f}(0, t) = 0$. In quantum mechanics on the contrary, $\mathbf{E}_{\perp f}$ is an operator. Although its average value can be zero (in the vacuum state for example), its quadratic average value is always strictly positive. The modifications of the electron dynamics originating from this term correspond to the effect of vacuum fluctuations.

To summarize, it is possible to derive a quantum form of the Abraham-Lorentz equation. The self reaction terms appear in a natural and unambiguous way and are formally identical in quantum and classical theories. In the 'quantum equation, the term describing the interaction of the particle with

(^{2}) As in classical theory, the fact that the mass correction in the Abraham-Lorentz equation and the mass correction in the rest mass energy (2.9) differ by a factor $\frac{4}{3}$ is due to the lack of covariance of the cut-off procedure.

the free field operator cannot be considered as a c-number equal to zero in the vacuum. We discuss now some consequences of the quantum nature of this last term.

2.3 THE DIFFICULTIES OF THE QUANTUM DYNAMICAL EQUATION. — In its traditional form, the classical Abraham-Lorentz equation suffers from a well known defect : the existence of preacceleration and self accelerated solutions. The discussion of the same problem in quantum theory is undoubtly interesting [18]. We prefer here to focus on some more fundamental difficulties inherent in the quantum formalism and which are hidden behind the formal analogy between the classical and the quantum dynamical equations.

First, it is worth noting that equation (2.24) relates non commuting operators. This of course complicates the resolution of the equation, but is unavoidable in a quantum theory of the electron dynamics.

Another difficulty lies in the fact that such an equation includes both particle and field operators, respectively $\mathbf{r}$, $\mathbf{p}$ and $\mathbf{E}_f$. This problem does not appear in the classical treatment where the free field, taken equal to zero, does not contribute to the Abraham-Lorentz equation. In quantum mechanics, $\mathbf{E}_f$ cannot be cancelled in the same way : physically, this means that the electron cannot escape the vacuum fluctuations. To estimate the two contributions of vacuum fluctuations and self reaction, we then have to integrate the quantum Abraham-Lorentz equation with a source term; this introduces further complications. To avoid this problem, one may try to deal only with electron operators averaged over the state of the field. Suppose that the radiation field is in the vacuum state at the initial time t_0 : Let $\langle S(t) \rangle_R$ be the average in this radiation state of the particle operator $S(t)$. $\langle S(t) \rangle_R$ is still an operator, acting

only in the electron Hilbert space. The average of equation (2.24) gives :

$$m \langle \ddot{\mathbf{r}} \rangle_{\mathbf{R}} = - \langle \nabla V_0(\mathbf{r}) \rangle_{\mathbf{R}} - \frac{4}{3} \delta m_1 \langle \ddot{\mathbf{r}} \rangle_{\mathbf{R}} +$$
$$+ \frac{2}{3} \frac{e^2}{4 \pi \varepsilon_0 c^3} \langle \dddot{\mathbf{r}} \rangle_{\mathbf{R}} . \quad (2.25)$$

We have used the fact that the average value of $\mathbf{E}_f$ is zero in the vacuum state. It seems in this last equation that vacuum fluctuations have disappeared and do not play any role in the evolution of $\langle \mathbf{r} \rangle_{\mathbf{R}}$. Actually, the simplicity of equation (2.25) is misleading; the averaged operators $\langle \mathbf{r} \rangle_{\mathbf{R}}$, $\langle \mathbf{p} \rangle_{\mathbf{R}}$ do not have the same properties as the original operators $\mathbf{r}$, $\mathbf{p}$. For example, their commutation relations are not the canonical ones ($[\langle \mathbf{r} \rangle_{\mathbf{R}}, \langle \mathbf{p} \rangle_{\mathbf{R}}] \neq i\hbar$) and their evolution is not unitary. So, we are no longer able to draw a parallel between the classical Abraham-Lorentz equation and the evolution of $\langle \mathbf{r} \rangle_{\mathbf{R}}$ given by (2.25).

Furthermore, all the dynamical aspects of the electron motion cannot be described only by the two operators $\langle \mathbf{r} \rangle_{\mathbf{R}}$ and $\langle \mathbf{p} \rangle_{\mathbf{R}}$. The value of the product $\langle \mathbf{r}.\mathbf{p} \rangle_{\mathbf{R}}$, for example, cannot be calculated as a function of $\langle \mathbf{r} \rangle_{\mathbf{R}}$ and $\langle \mathbf{p} \rangle_{\mathbf{R}}$. Similarly, equation (2.25) is not a true differential equation since $\langle \nabla V_0(\mathbf{r}) \rangle_{\mathbf{R}}$ cannot be expressed in terms of $\langle \mathbf{r} \rangle_{\mathbf{R}}$ and $\langle \mathbf{p} \rangle_{\mathbf{R}}$. This equation is then not « closed » : it links $\langle \mathbf{r} \rangle_{\mathbf{R}}$ and its derivatives to another operator $\langle \nabla V_0(\mathbf{r}) \rangle_{\mathbf{R}}$ for which we have to find the evolution equation (the vacuum fluctuations will probably contribute to this equation, which proves that their disappearance in (2.25) was only superficial).

The previous discussion clearly shows that we cannot avoid to study now the evolution of electron observables other than $\mathbf{r}$ and $\mathbf{p}$ and to ask about their rate of variation the same type of questions concerning the respective contributions of vacuum fluctuations and self reaction. This problem will be be dealt with in the next section. Note that the simplifications which occurred above for the evolution of $\mathbf{r}$ (no order problem in (2.24) and nullity of the vacuum average of $\mathbf{E}_f$ in (2.25)) will not occur for the evolution of a general particle observable.

There is a supplementary reason for studying the evolution of operators other than $\mathbf{r}$ and $\mathbf{p}$. Very few experiments are dealing with the position or the momentum of an atomic electron. One rather measures the population of an energy level, the frequency or the damping of some atomic oscillations associated with off-diagonal elements of the density matrix. This suggests that operators such as $| i \rangle \langle i |$ or $| i \rangle \langle j |$ (where $| i \rangle$ and $| j \rangle$ are eigenstates of the electron in the potential V_0) are more directly connected to experiment than $\mathbf{r}$ and $\mathbf{p}$.

3. Identification of the contributions of vacuum fluctuations and self reaction to the rate of variation of an arbitrary atomic observable. — In this section,

we first evaluate the contributions of the various terms of the interaction Hamiltonian to the rate of variation, dG/dt, of an arbitrary atomic observable G (§ 3.1). We then discuss the problem of order which arises when the total field appearing in this rate is split into its free part and its source part (§ 3.2). We solve this problem by introducing hermiticity conditions associated with the requirement of physical meaning (§§ 3.3 and 3.4). Finally, we discuss the problem of the vacuum average of the various rates which requires a perturbative calculation (§ 3.5).

3.1 CONTRIBUTION OF THE VARIOUS TERMS OF THE INTERACTION HAMILTONIAN. — It will be convenient to divide the total Hamiltonian (2.13) into three parts, the Hamiltonian

$$H_s = \frac{\mathbf{p}^2}{2m} + V_0(\mathbf{r}) \quad (3.1)$$

of the electron in the static potential $V_0(\mathbf{r})$, the Hamiltonian

$$H_{\mathbf{R}} = \sum_{\mathbf{k}\varepsilon} \hbar\omega \left(a_{\mathbf{k}\varepsilon}^+ a_{\mathbf{k}\varepsilon} + \frac{1}{2} \right) \quad (3.2)$$

of the transverse radiation field, and the Hamiltonian

$$V = - \frac{e}{m} \mathbf{p}.\mathbf{A}(0) + \frac{e^2}{2m} \mathbf{A}^2(0) + \frac{e^2}{4\pi\varepsilon_0} \frac{k_M}{\pi} \quad (3.3)$$

of the electron-field coupling, characterized by the electric charge e and including the energy of the longitudinal field of the electron (2.12).

The rate of variation of an atomic observable G can then be written as

$$\frac{d}{dt} G = \frac{i}{\hbar} [H_s, G] + \frac{i}{\hbar} [V, G] . \quad (3.4)$$

We discuss now the contributions of the three terms of V to the second commutator (to order 2 in e).

(i) The last term of V is a c-number which commutes with G and which therefore does not produce any dynamical evolution. This term corresponds to an overall displacement of electronic energy levels which we have already interpreted in section 2 as due to the contribution $\delta m_1 c^2$ of the Coulomb field of the electron to the electron rest mass energy. This effect must obviously be associated with self reaction since it originates from the longitudinal field created by the electron itself. The same situation exists in classical theory.

(ii) The second term of V does not depend on atomic variables and thus commutes with G. It has no dynamical consequences. It nevertheless contributes to the total energy. Let us calculate its average value. Since we limit the calculation to order 2 in e, we can replace $\mathbf{A}(0)$ by the free field $\mathbf{A}_f(0)$. The term then becomes independent of the atomic state and can be interpreted as an overall shift of the electron

energy levels. The value of this shift for the vacuum state of the field is given by

$$\left\langle 0 \left| \frac{e^2}{2m} A_f^2(0) \right| 0 \right\rangle = \sum_{k\epsilon} \frac{e^2 \mathcal{A}_{k^2}}{2m} = \frac{e^2 \hbar k_M^2}{8 \pi^2 \epsilon_0 mc} =$$

$$= \delta m_2 c^2 . \quad (3.5)$$

This shift can be interpreted as a new contribution, $\delta m_2 c^2$, to the electron rest mass energy. It is proportional to the vacuum average of the square of the free field and thus is clearly a vacuum fluctuation effect, the interpretation of which is well known [19] : it is the kinetic energy associated with the electron vibrations produced by the vacuum fluctuations of the electric field.

(iii) Finally, only the first term of (3.3) contributes to the dynamical evolution of G. The corresponding term of (3.4) can be written as

$$\left(\frac{dG}{dt}\right)_{\text{coupling}} = -\frac{ie}{\hbar m} [\mathbf{p}.\mathbf{A}(0), G] = e\mathbf{N}.\mathbf{A}(0) \quad (3.6)$$

where $\mathbf{N}$ is an atomic operator given by

$$\mathbf{N} = -\frac{i}{\hbar m} [\mathbf{p}, G] . \quad (3.7)$$

If G coincides with $\mathbf{p}$ or $\mathbf{r}$, $\mathbf{N}$ is equal to $\mathbf{0}$ or to a constant and (3.6) reduces to $\mathbf{0}$ or to $\mathbf{A}(0)$. We find again that the evolution of $\mathbf{r}$ and $\mathbf{p}$ is very simple.

Finally, combining (3.4) and (3.6) and reintroducing the time explicitly in the operators, we get

$$\frac{d}{dt} G(t) = \frac{i}{\hbar} [H_s(t), G(t)] + e\mathbf{N}(t).\mathbf{A}(0, t) . \quad (3.8)$$

3.2 THE PROBLEM OF ORDER.

— In expression (3.8), we split, as in section 2, the field $\mathbf{A}(0, t)$ in two parts, $\mathbf{A}_f(0, t)$ representing the free field and $\mathbf{A}_s(0, t)$ representing the source field. If the atomic operator $\mathbf{N}(t)$ does not reduce to $\mathbf{0}$ or to a constant (as it is the case for $\mathbf{r}$ and $\mathbf{p}$), we are immediately faced with the problem of order mentioned in the introduction. Since $\mathbf{N}(t)$ and $\mathbf{A}(0, t)$ commute, we can start in equation (3.8) with any order

$$\mathbf{N}(t).\mathbf{A}(0, t) \quad \text{or} \quad \mathbf{A}(0, t).\mathbf{N}(t) .$$

More generally, we can write the last term of (3.8) as

$$e\lambda\mathbf{N}(t).\mathbf{A}(0, t) + e(1 - \lambda) \mathbf{A}(0, t).\mathbf{N}(t) \quad (3.9)$$

with λ arbitrary. Replacing $\mathbf{A}$ by $\mathbf{A}_f + \mathbf{A}_s$ leads to

$$\left(\frac{dG}{dt}\right)_{\text{coupling}} = \left(\frac{dG}{dt}\right)_{\text{vf}} + \left(\frac{dG}{dt}\right)_{\text{sr}} \quad (3.10)$$

where the two rates

$$\left(\frac{dG}{dt}\right)_{\text{vf}} = e\lambda\mathbf{N}(t).\mathbf{A}_f(0, t) + e(1 - \lambda) \mathbf{A}_f(0, t).\mathbf{N}(t)$$

$$(3.11)$$

$$\left(\frac{dG}{dt}\right)_{\text{sr}} = e\lambda\mathbf{N}(t).\mathbf{A}_s(0, t) + e(1 - \lambda) \mathbf{A}_s(0, t).\mathbf{N}(t)$$

$$(3.12)$$

depend on λ since $\mathbf{A}_f$ and $\mathbf{A}_s$ do not commute separately with $\mathbf{N}(t)$.

λ being arbitrary, the splitting (3.10) of the total rate is not uniquely defined [11-13].

3.3 PHYSICAL INTERPRETATION AND HERMITICITY CONDITIONS.

— In order to remove this indetermination, we introduce now some simple physical considerations.

Suppose that G is a physical observable, represented by a Hermitian operator. The rate of variation of G due to the coupling is also a Hermitian operator [this clearly appears on (3.8) since $\mathbf{N}(t)$ and $\mathbf{A}(0, t)$ are commuting Hermitian operators]. Our purpose is to split this rate of variation in two rates, involving $\mathbf{A}_f$ and $\mathbf{A}_s$ respectively, and having separately a well defined physical interpretation in terms of vacuum fluctuations and self reaction. This interpretation requires that (3.11) and (3.12) should have separately a physical meaning, and consequently should be separately Hermitian. This condition determines λ which must be equal to 1/2. Thus, the splitting of dG/dt is unique and given by

$$\left(\frac{dG}{dt}\right)_{\text{vf}} = e\frac{1}{2} [\mathbf{N}(t).\mathbf{A}_f(0, t) + \mathbf{A}_f(0, t).\mathbf{N}(t)]$$

$$(3.13)$$

$$\left(\frac{dG}{dt}\right)_{\text{sr}} = e\frac{1}{2} [\mathbf{N}(t).\mathbf{A}_s(0, t) + \mathbf{A}_s(0, t).\mathbf{N}(t)] .$$

$$(3.14)$$

This could have been obtained by choosing the completely symmetrical order in (3.9).

3.4 GENERALIZATION TO MORE COMPLICATED SITUATIONS.

— It may happen that the total rate of variation of G does not appear as simple as in (3.6), i.e. as the product of an atomic observable by a field observable. For example, if we had not made the electric dipole approximation, the electron position operator $\mathbf{r}$ would appear in $\mathbf{A}$. Another example is the appearance of non Hermitian operators in (3.6) when the total field $\mathbf{A}$ is decomposed into its positive and negative frequency components which are not Hermitian. We extend now the previous treatment to these more complex situations.

We first note that, in the most general case, the total rate of variation of a physical observable G

(due to the coupling with the field) can always be written as

$$\left(\frac{dG}{dt}\right)_{\text{coupling}} = \sum_i e(A_i \, N_i + A_i^+ \, N_i^+) \quad (3.15)$$

where the A_i are field operators and the N_i atomic operators which commute, but which are not necessarily Hermitian. For example, in simple models dealing with two-level atoms and using the « rotating wave approximation », the coupling Hamiltonian is taken of the form

$$V = - (E^{(+)} D^+ + E^{(-)} D^-) \quad (3.16)$$

where D^+ and D^- are the raising and lowering components of the dipole moment operator, and $E^{(+)}$ and $E^{(-)}$ the positive and negative frequency components of the field [20].

In such a case, we get

$$\left(\frac{dG}{dt}\right)_{\text{coupling}} = E^{(+)} F^+ + E^{(-)} F^- \quad (3.17)$$

with .

$$F^+ = \frac{i}{\hbar} [G, D^+] \quad (3.18)$$

(3.17) has a structure similar to (3.15).

G being Hermitian, the right side of (3.15) is of course also Hermitian, but since the atomic and field operators commute, it could be written as well as

$$\sum_i (N_i \, A_i + N_i^+ \, A_i^+) \quad \text{or} \quad \sum_i (N_i \, A_i + A_i^+ \, N_i^+)$$

or

$$\sum_i (A_i \, N_i + N_i^+ \, A_i^+)$$

or any combination of these forms. When A_i is replaced by $A_{if} + A_{is}$, it is easy to see that the hermiticity condition imposed on $\left(\frac{dG}{dt}\right)_{\text{vf}}$ and $\left(\frac{dG}{dt}\right)_{\text{sr}}$ is no longer sufficient for removing the indetermination. For example,

$$\sum_i (A_{if} \, N_i + N_i^+ \, A_{if}^+) \quad \text{and} \quad \sum_i (N_i \, A_{if} + A_{if}^+ \, N_i^+)$$

are two Hermitian rates of variation which could be attributed to vacuum fluctuations and which generally do not coincide since A_{if} and A_{if}^+ do not commute with N_i and N_i^+. For the simple model considered above [see (3.16) and (3.17)] these two rates respectively correspond to the anti normal and normal orders. So, when the A_i and the N_i are not Hermitian, we must introduce a new requirement.

Coming back to the expression (3.15) of the total rate, we first re-express this rate in terms of physically well defined atomic and field quantities, i.e. in terms of Hermitian operators. The physical justification for such a transformation is that we want to be able to analyse the total physical rate in terms of physical quantities. For example, it would be difficult to elaborate a physical picture from an expression involving only the positive frequency part of the field which is not observable. Introducing the real and the imaginary part of the various operators appearing in (3.15), and using the fact that field and atomic operators commute, we transform (3.15) into the strictly equivalent expression

$$\left(\frac{dG}{dt}\right)_{\text{coupling}} = e \sum_i \left(\frac{A_i + A_i^+}{2}\right)(N_i + N_i^+) + e \sum_i \left(\frac{A_i - A_i^+}{2 \, i}\right)\left(\frac{N_i - N_i^+}{- \, i}\right). \quad (3.19)$$

But now the total rate appears as a sum of products of observables of the field by observables of the particle as in (3.6) and the procedure of the previous section can be applied to each of these products and singles out the completely symmetric order

$$\left(\frac{dG}{dt}\right)_{\text{coupling}} = e \frac{1}{2} \sum_i \left[\left(\frac{A_i + A_i^+}{2}\right)(N_i + N_i^+) + (N_i + N_i^+)\left(\frac{A_i + A_i^+}{2}\right)\right] +$$

$$+ \, e \frac{1}{2} \sum_i \left[\left(\frac{A_i - A_i^+}{2 \, i}\right)\left(\frac{N_i - N_i^+}{- \, i}\right) + \left(\frac{N_i - N_i^+}{- \, i}\right)\left(\frac{A_i - A_i^+}{2 \, i}\right)\right] \quad (3.20)$$

when A_i is replaced by $A_{if} + A_{is}$ in (3.19).

To summarize the previous discussion, a unique well defined order is singled out by the following two conditions.

(i) The two rates $\left(\frac{dG}{dt}\right)_{\text{vf}}$ and $\left(\frac{dG}{dt}\right)_{\text{sr}}$ must have separately a physical meaning.

(ii) Before replacing A_i by $A_{if} + A_{is}$, the total rate must be expressed in terms of physical field and particle quantities.

3.5 VACUUM AVERAGE OF THE VARIOUS RATES. —
To progress further, we must now take the average of the two rates (3.13) and (3.14) over the vacuum

state of the field. The calculation of such an average is not trivial (as it was the case in the previous section for **r** and **p**). This is due to the presence of products of field and atomic operators in the right side of the equations. For example, when we average the product $e\mathbf{A}_f(0, t).\mathbf{N}(t)$, we must not forget that these two operators are correlated since the atomic operator $\mathbf{N}(t)$ depends on the free field which has perturbed its evolution from the initial time t_0 to t. Consequently, before taking the vacuum average, we have first to calculate, to a given order in e, $\mathbf{N}(t)$ as a function of unperturbed (free) atomic and field operators. Since we limit our calculation to order 2 in e (i.e. to order 1 in the fine structure constant α), we must solve the Heisenberg equation for $\mathbf{N}(t)$ up to order e [e already appears in (3.13) and $\mathbf{A}_f$ is of order e^0]. When we insert the perturbative expansion of $\mathbf{N}(t)$, which contains zero or one field operator taken at a time t' such that $t_0 < t' < t$, in the product $\mathbf{A}_f(0, t).\mathbf{N}(t)$, and when we take the vacuum average, we get « one-time averages » $\langle 0 \,|\, \mathbf{A}_f(t) \,|\, 0 \rangle$ which are equal to zero, and « two-time averages » such as

$$\langle 0 \,|\, A_{fi}(0, t) \, A_{fj}(0, t') \,|\, 0 \rangle$$

(with $i, j = x, y, z$), i.e. vacuum averages of products of two components of free field operators taken at two different times. Similar considerations can be made about the other products of (3.13) and (3.14).

Actually, such perturbative calculations are not specific of our choice of the symmetrical order in (3.9) and they can be found in other papers where other choices are investigated [11, 13]. Rather than duplicating these calculations, we prefer in the next section to reconsider our problem of the separation between vacuum fluctuations and self reaction from a more general point of view where one asks the same type of questions for a small system S (generalizing the atom) interacting with a large reservoir $\mathfrak{R}$ (generalizing the field). The extension of the previous treatment to this more general situation is straightforward. It leads to mathematical expressions which, because of their generality, have a more transparent structure. In particular, since we don't use, in the intermediate steps of the calculation, simplifications specific to a particular choice of S and $\mathfrak{R}$, we find that some important statistical functions of S and $\mathfrak{R}$ appear explicitly in the final expressions and this provides a deeper physical insight in the problem.

4. Extension of the previous treatment to a system S interacting with a large reservoir $\mathfrak{R}$. — 4.1 INTRODUCTION-OUTLINE OF THE CALCULATION. — It is well known that spontaneous emission, and all associated effects such as radiative corrections or radiative damping, can be considered as a problem which can be studied in the general framework of the quantum theory of relaxation in the motional narrowing limit [21, 22]. Such a theory deals with the damping and energy shifts of a small system S coupled to a large

reservoir $\mathfrak{R}$. Large means that $\mathfrak{R}$ has many degrees of freedom so that the correlation time τ_c of the observables of $\mathfrak{R}$ is very short, allowing a perturbative treatment of the effect of the S-$\mathfrak{R}$ coupling during a time τ_c. For spontaneous emission, the atom plays the role of S, the vacuum field, with its infinite number of modes, plays the role of $\mathfrak{R}$, and the correlation time of vacuum fluctuations is short enough for having the motional narrowing condition well fulfilled.

This point of view suggests that we can extend to any S-$\mathfrak{R}$ system the same type of questions we have asked about the atom-field system. Is it possible to undestand the evolution of S as being due to the effect of the reservoir fluctuations acting upon S, or should we invoke a kind of self reaction, S perturbing $\mathfrak{R}$ which reacts back on S ? Is it possible to make a clear and unambiguous separation between the contributions of these two effects ?

The extension of the treatment of section 3 to this more general case is straightforward. We first note that, although most presentations of the quantum theory of relaxation use the Schrödinger picture (one derives a master equation for the reduced density operator of S), we have to work here in the Heisenberg picture. Actually, the Heisenberg picture is also used in the derivation of the « Langevin-Mori » equations describing the evolution of the observables of S as being driven by a « Langevin force » (having a zero reservoir average) and a « friction force » (producing not only a damping but also a shift of energy levels) [21, 23, 24]. Our problem here is to identify in the « friction force » the contribution of reservoir fluctuations and self reaction. Following the general scheme of figure 1, we start with the Hamiltonian of the S-$\mathfrak{R}$ system

$$H = H_s + H_R + V \tag{4.1}$$

where

$$V = - \sum_i R_i S_i \tag{4.2}$$

is the interaction Hamiltonian, and R_i and S_i are Hermitian observables of $\mathfrak{R}$ and S [we can always suppose that V has been put in this form, eventually after a transformation analogous to the one changing (3.15) into (3.19)]. We then write the Heisenberg equation for the reservoir observable R_i appearing in (4.2). The solution of this equation can be written as the sum of a free unperturbed part R_{if} (solution to order 0 in V), and of a « source part » R_{is} due to the presence of S (solution to order 1 and higher in V)

$$R_i = R_{if} + R_{is} . \tag{4.3}$$

Expression (4.3) is finally inserted in the last term of the Heisenberg equation for an arbitrary system observable G

$$\frac{dG}{dt} = \frac{1}{i\hbar} [G, H_s] + \frac{1}{i\hbar} \left[G, - \sum_i R_i S_i \right] \tag{4.4}$$

in order to indentify the contribution of reservoir fluctuations and self reaction. The problem of order between the commuting observables R_i and

$$N_i = \frac{-1}{i\hbar} [G, S_i] \qquad (4.5)$$

in the last term of (4.4) arises in the same way as in section 3 and is solved by the same physical considerations which impose the completely symmetric order. We thus get

$$\left\{ \begin{array}{l} \left(\dfrac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{rf}} = \dfrac{1}{2}\sum_i (N_i\, R_{\mathrm{if}} + R_{\mathrm{if}}\, N_i) \qquad (4.6a) \\[4mm] \left(\dfrac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{sr}} = \dfrac{1}{2}\sum_i (N_i\, R_{\mathrm{is}} + R_{\mathrm{is}}\, N_i). \qquad (4.6b) \end{array} \right.$$

It remains to perform the average of $\left(\dfrac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{rf}}$ and $\left(\dfrac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{sr}}$ in the reservoir state (reservoir average). As explained in § 3.5, this requires a perturbative calculation leading, to order 2 in V, to two time operator averages which can be expressed in terms of correlation functions and linear susceptibilities. This is precisely where the advantage of working with a general S-R system appears. As already explained in § 3.5, the intermediate steps of the calculation remain general. For example, when we solve perturbatively the Heisenberg equation for R_i, we get for the source part, R_{is}, a perturbative expansion where, at the lowest order, the linear susceptibility of the reservoir appears. In the particular case of the atom field system, the calculation of the source field has been done exactly and the result expressed in terms of atomic operators and time derivatives of these operators (see equation (2.21)). In such an intermediate calculation, the fact that the susceptibility of the electromagnetic field is involved remains hidden, and, thus this important function does not appear explicitly in the final result for $\left\langle \left(\dfrac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{sr}} \right\rangle$.

In order not to increase too much the length of this paper, we will not give here the detailed calculations of $\left\langle \left(\dfrac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{rf}} \right\rangle$ and $\left\langle \left(\dfrac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{sr}} \right\rangle$ following the general scheme outlined above. These calculations will be presented in a forthcoming paper [25], together with a discussion of the various approximations used in the derivation. We just give in this section the results of these calculations which will be useful for the discussion of section 5. We first give (§ 4.2) the expression of the correlation functions and linear susceptibilities in terms of which we then discuss the structure of the terms describing the effect of reservoir fluctuations (§ 4.3) and self reaction (§ 4.4).

4.2 CORRELATION FUNCTIONS AND LINEAR SUSCEPTIBILITIES [26]. — When the reservoir average is cal-

culated up to order 2 in V, the reservoir only appears in the final result through two statistical functions. The first one

$$C_{ij}^{(\mathrm{R})}(\tau) =$$
$$= \frac{1}{2} \langle \{ R_{\mathrm{if}}(t)\, R_{\mathrm{jf}}(t - \tau) + R_{\mathrm{jf}}(t - \tau)\, R_{\mathrm{if}}(t) \} \rangle_{\mathrm{R}}$$
$$\qquad (4.7)$$

is the symmetric correlation function of the two free reservoir observables R_{if} and R_{jf}. The average is taken over the initial state of the reservoir which is supposed to be stationary, so that $C_{ij}^{(\mathrm{R})}$ only depends on τ. $C_{ij}^{(\mathrm{R})}(\tau)$ is a real function of τ which describes the dynamics of the fluctuations of R_{if} and R_{jf} in the reservoir state.

The second statistical function,

$$\chi_{ij}^{(\mathrm{R})}(\tau) = \frac{i}{\hbar} \langle [R_{\mathrm{if}}(t), R_{\mathrm{jf}}(t - \tau)] \rangle_{\mathrm{R}} \theta(\tau), \qquad (4.8)$$

where $\theta(\tau)$ is the Heaviside function, is the linear susceptibility of the reservoir. It generally depends on the reservoir state. $\chi_{ij}^{(\mathrm{R})}(\tau)$ is also a real function of τ, which describes the linear response of the averaged observable $\langle R_{\mathrm{if}}(t) \rangle_{\mathrm{R}}$ when the reservoir is acted upon by a perturbation proportional to R_{jf}. Note that both C and χ have a classical limit (if it the case for $\mathcal{R}$) : this is obvious for C, and for χ, the commutator divided by $i\hbar$ becomes the Poisson bracket.

Similar functions can of course be introduced for the small system S in an energy level $| a \rangle$, with energy E_a. We will note them

$$C_{ij}^{(\mathrm{S},a)}(\tau) = \frac{1}{2} \langle a \mid S_{\mathrm{if}}(t)\, S_{\mathrm{jf}}(t - \tau) + S_{\mathrm{jf}}(t - \tau)\, S_{\mathrm{if}}(t) \mid a \rangle \qquad (4.9)$$

$$\chi_{ij}^{(\mathrm{S},a)}(\tau) = \frac{i}{\hbar} \langle a \mid [S_{\mathrm{if}}(t), S_{\mathrm{jf}}(t - \tau)] \mid a \rangle \theta(\tau) \qquad (4.10)$$

where the upper indices (S, a) mean that S is in $| a \rangle$, and where the lower index f on S_{if} and S_{jf} means that these operators are unperturbed system operators evolving only under the effect of H_{s} (as for R_{if} and R_{jf} which evolve only under the effect of H_{R}).

Finally, we will note $\hat{C}_{ij}^{(\mathrm{R})}(\omega)$, $\hat{\chi}_{ij}^{(\mathrm{R})}(\omega)$, $\hat{C}_{ij}^{(\mathrm{S},a)}(\omega)$, $\hat{\chi}_{ij}^{(\mathrm{S},a)}(\omega)$ the Fourier transforms of (4.7), (4.8), (4.9), (4.10), the Fourier transform $\hat{f}(\omega)$ of $f(\tau)$ being defined by

$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\tau)\, \mathrm{e}^{-i\omega\tau}\, \mathrm{d}\tau. \qquad (4.11)$$

4.3 STRUCTURE OF THE RESULTS DESCRIBING THE EFFECT OF RESERVOIR FLUCTUATIONS. — The first important result concerning the reservoir averaged rate of variation $\left\langle \left(\dfrac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{rf}} \right\rangle_{\mathrm{R}}$ is that only $C_{ij}^{(\mathrm{R})}(\tau)$

appears in its expression, and not $\chi_{ij}^{(R)}(\tau)$. Furthermore, the corresponding relaxation equations have exactly the same structure as the ones which would be obtained if the reservoir observables R_i were replaced in the interaction Hamiltonian (4.2) by fluctuating c-numbers $r_i(t)$ having the same correlation functions $C_{ij}^{(R)}(\tau)$

$$\overline{r_i(t)\, r_j(t - \tau)} = C_{ij}^{(R)}(\tau). \qquad (4.12)$$

We conclude that, with our choice of the symmetric order in (4.6), the effect of reservoir fluctuations is the same as the one of a classical random field having the same symmetric correlation function as the quantum one.

We show also in reference [25] that the average rate of variation $\left\langle \left(\dfrac{dG}{dt}\right)_{rf} \right\rangle_R$, and also $\left\langle \left(\dfrac{dG}{dt}\right)_{sr} \right\rangle_R$, can be decomposed into a Hamiltonian part and a non Hamiltonian part. The Hamiltonian part describes (in the so-called secular approximation) a shift of the energy levels of S due to the S-$\Re$ coupling. The non Hamiltonian part describes, among other things, the exchange of energy between S and $\Re$.

The shift, $(\delta E_a)_{rf}$, of the level $| a \rangle$ of S due to reservoir fluctuations is found to be

$$(\delta E_a)_{rf} = -\frac{1}{2} \sum_{ij} \int_{-\infty}^{+\infty} d\tau \; C_{ij}^{(R)}(\tau)\, \chi_{ij}^{(S,a)}(\tau). \quad (4.13)$$

Such a result has a very simple structure and a very clear physical meaning (Fig. 2a). One can consider that the fluctuations of $\Re$, characterized by $C_{ij}^{(R)}(\tau)$, polarize S which responds to this perturbation in a way characterized by $\chi_{ij}^{(S,a)}(\tau)$. The interaction of the fluctuations of $\Re$ with the polarization to which they give rise in S has a non zero value because of the correlations which exist between the fluctuations of $\Re$ and the induced polarization in S. The factor $1/2$ in (4.13) is even somewhat similar to the factor $1/2$ appearing in the polarization energy of a dielectric. Finally, it is shown in [25] (by parity arguments) that only the reactive part of $\chi_{ij}^{(S,a)}(\tau)$ contributes to the
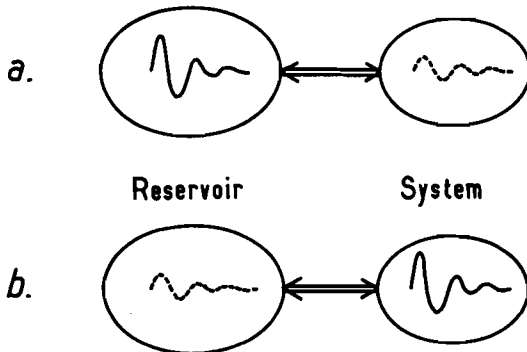

Fig. 2. — Physical pictures for the effect of reservoir fluctuations and self reaction. a) Reservoir fluctuations : the reservoir fluctuates and interacts with the polarization induced in the small system. b) Self reaction : the small system fluctuates and polarizes the reservoir which reacts back on the small system.

integral (4.13). To summarize this discussion, we can say that the energy shift $(\delta E_a)_{rf}$ can be interpreted as resulting from the polarization of S by the fluctuations of $\Re$.

We now turn to the discussion of the non Hamiltonian part of $\left\langle \left(\dfrac{dG}{dt}\right)_{rf} \right\rangle_R$. A very suggestive result concerns the absorption of energy by S when S is in $| a \rangle$. The effect is described by $\left\langle \left(\dfrac{dH_s}{dt}\right)_{rf} \right\rangle_{R,a}$ (G is replaced by H_s and the average is taken over both the state of the reservoir and the state $| a \rangle$ of S). One finds

$$\left\langle \left(\frac{dH_s}{dt}\right)_{rf} \right\rangle_{R,a} =$$
$$= -\pi \sum_{ij} \int_{-\gamma}^{+\infty} d\omega \; \hat{C}_{ij}^{(R)}(\omega)\, i\omega[\hat{\chi}_{ij}^{(S,a)*}(\omega) - \hat{\chi}_{ij}^{(S,a)}(\omega)].$$
$$(4.14)$$

This result is identical with the one which would be obtained if a classical random perturbation with a spectral power density $\hat{C}_{ij}^{(R)}(\omega)$ was acting upon S (see reference [27], § 124; see also [28]). The term inside the brackets is actually the dissipative part of the susceptibility of S at frequency ω. This dissipative part is multiplied by the spectral power density of the perturbation produced by $\Re$. Here again we get a result in agreement with the picture of S responding to the fluctuations of $\Re$.

4.4 STRUCTURE OF THE TERMS DESCRIBING THE EFFECT OF SELF REACTION. —As expected, the reservoir appears in $\left\langle \left(\dfrac{dG}{dt}\right)_{sr} \right\rangle_R$ only through the linear susceptibility $\chi_{ij}^{(R)}(\tau)$. Thus, it appears that $\Re$ is now polarized by S. We can interpret the rate of variation $\left\langle \left(\dfrac{dG}{dt}\right)_{sr} \right\rangle_R$ as being due to the reaction back on S of the polarization of $\Re$ by S (Fig. 2b).

As in the previous section (4.3), it will be interesting now to discuss the shift $(\delta E_a)_{sr}$ of $| a \rangle$ due to self reaction. This shift is found to be

$$(\delta E_a)_{sr} = -\frac{1}{2} \sum_{ij} \int_{-\infty}^{+\infty} d\tau \; \chi_{ij}^{(R)}(\tau)\, C_{ij}^{(S,a)}(\tau). \quad (4.15)$$

The same comments can be made as for (4.13), the roles of S and $\Re$ being interchanged. Here also, only the reactive part of $\chi_{ij}^{(R)}(\tau)$ contributes to (4.15).

Finally, we can study the equation corresponding to (4.14) for self reaction

$$\left\langle \left(\frac{dH_s}{dt}\right)_{sr} \right\rangle_{R,a} =$$
$$= \pi \sum_{ij} \int_{-\infty}^{+\infty} d\omega \; \hat{C}_{ij}^{(S,a)}(\omega)\, i\omega[\hat{\chi}_{ij}^{(R)*}(\omega) - \hat{\chi}_{ji}^{(R)}(\omega)].$$
$$(4.16)$$

Here also the same comments can be made, the roles of S and $\mathcal{R}$ being interchanged. Note however the difference of sign between (4.14) and (4.16). This is due to the fact that (4.14) describes a transfer of energy from $\mathcal{R}$ to S (gain for S), whereas (4.16) describes a transfer from S to $\mathcal{R}$ (loss for S). Actually (4.14) can also describe a loss for S, and (4.16) a gain, if there are adequate population inversions in S for (4.14), in $\mathcal{R}$ for (4.16), responsible for an amplifying behaviour of the susceptibility (instead of a dissipative one).

It must be emphasized that all the results derived in this section follow from the choice of the symmetric order in the total rate dG/dt before replacing R_i by $R_{if} + R_{is}$. They can be all interpreted in terms of two simple physical pictures : $\mathcal{R}$ fluctuates and polarizes S, S fluctuates and polarizes $\mathcal{R}$. The clear physical structure of the results which have been obtained in this way, and the coherence of the physical interpretation can be considered as a confirmation *a posteriori* of the pertinence of the method of separation we propose in this paper. The priviliged character of the symmetric order for physical interpretation is thus confirmed.

Remark. — The previous treatment allows an easy and clear discussion of the consequences of the fluctuation dissipation theorem [26]. Note first that this theorem holds only for systems in thermal equilibrium (populations of the various levels varying according to the Boltzmann factor corresponding to a given temperature). The above treatment is more general, and is valid for an arbitrary stationary state of the reservoir (the energy levels may have any population). For a reservoir at thermal equilibrium which is the case of the electromagnetic field vacuum, the fluctuation dissipation theorem states that the correlation function $\hat{C}_{ij}^{(R)}(\omega)$ is proportional to the dissipative part of the corresponding reservoir susceptibility. Thus, in this case, one could formally replace in (4.13) and (4.14) the correlation function of the reservoir by the dissipative part of the reservoir susceptibility

and make the reservoir fluctuations to apparently disappear from formulae (4.13) and (4.14). But it is also clear that, after such a formal transformation, these two expressions have lost their physical meaning since they appear as the product of two susceptibilities.

5. Physical discussion. Contributions of vacuum fluctuations and self reaction to the radiative corrections and radiative damping of an atomic electron. —

We now come back to our initial problem concerning the respective contributions of vacuum fluctuations and self reaction for an atomic electron.

We have given in the previous section very simple and general expressions for important physical effects such as the shifts of the energy levels of S, or the energy exchanges between S and $\mathcal{R}$, these expressions involving only correlation functions or linear susceptibilities of S and $\mathcal{R}$.

What we have to do now is to calculate first these correlation functions and linear susceptibilities in the case where S is an atom and $\mathcal{R}$ the vacuum electromagnetic field (§ 5.1). We will then be able, using (4.13), (4.14), (4.15), (4.16), to discuss the respective contributions of vacuum fluctuations and self reaction to the radiative corrections for an atomic electron (§ 5.2) and the rate of exchange of energy between the atom and the field (§ 5.3).

5.1 CORRELATION FUNCTIONS AND LINEAR SUSCEPTIBILITIES FOR THE VACUUM FIELD AND FOR AN ATOMIC ELECTRON. — Comparing (4.2) and the first term of (3.3) (which is the only one to produce a dynamical evolution of atomic observables, see § 3.1), we get, for the atom field problem

$$\begin{cases} R_i(t) = A_i(0, t) \\ S_i(t) = \dfrac{e}{m} p_i(t) \end{cases} \tag{5.1}$$

with $i = x, y, z$.

According to (4.7) and (4.8), the relevant statistical functions for the field are :

$$C_{ij}^{(R)}(\tau) = \frac{1}{2} \langle 0 \mid A_{if}(0, t) A_{jf}(0, t - \tau) + A_{jf}(0, t - \tau) A_{if}(0, t) \mid 0 \rangle \tag{5.2}$$

$$\chi_{ij}^{(R)}(\tau) = \frac{i}{\hbar} \langle 0 \mid [A_{if}(0, t), A_{jf}(0, t - \tau)] \mid 0 \rangle \theta(\tau) \tag{5.3}$$

where $\mid 0 \rangle$ is the vacuum state of the field and the index f means a free evolution for the operators. The calculation of these two functions is straightforward and given in the Appendix B. One gets :

$$C_{ij}^{(R)}(\tau) = \frac{\hbar \delta_{ij}}{12 \pi^2 \varepsilon_0 c^3} \int_{-\omega_M}^{\omega_M} \mid \omega \mid e^{i\omega\tau} d\omega \tag{5.4}$$

$$\chi_{ij}^{(R)}(\tau) = \frac{\delta_{ij}}{3 \pi^2 \varepsilon_0 c^3} \left(\omega_M \, \delta(\tau) - \frac{\pi}{2} \, \delta'(\tau) \right). \tag{5.5}$$

The Fourier transforms of (5.4) and (5.5) are also useful :

$$\hat{C}_{ij}^{(R)}(\omega) = \frac{\hbar \delta_{ij}}{12 \pi^2 \varepsilon_0 c^3} |\omega| \tag{5.6}$$

$$\hat{\chi}_{ij}^{(R)}(\omega) = \frac{\delta_{ij}}{6 \pi^3 \varepsilon_0 c^3} \left(\omega_M - i \frac{\pi}{2} \omega \right). \tag{5.7}$$

Because of the cut-off (2.10) expressions (5.6) and (5.7) hold only for $|\omega| < \omega_M$, $\hat{C}$ and $\hat{\chi}$ being equal to zero elsewhere. It follows that the δ and δ' functions in (5.5) have actually a width $1/\omega_M$.

Remarks : (i) The linear susceptibility of the field relates the linear response of the field, at point **0** and at time *t*, to the perturbation associated with the motion of the electron at earlier times. This response is nothing but the source field produced by the electron (and calculated to lowest order in *e*). Going back to the precise definition of χ [26], and using (5.5), we get for the « linear response » $\langle 0 | A_i(t) | 0 \rangle$:

$$\langle 0 | A_i(t) | 0 \rangle = \sum_j \int_{-\infty}^{+\infty} d\tau \left\langle 0 \left| \chi_{ij}^{(R)}(\tau) \left(\frac{e}{m} \right) p_j(t - \tau) \right| 0 \right\rangle$$

$$= \frac{e\omega_M}{3 \pi^2 \varepsilon_0 c^3 m} \langle 0 | p_i(t) | 0 \rangle - \frac{e}{6 \pi \varepsilon_0 c^3 m} \langle 0 | \dot{p}_i(t) | 0 \rangle \tag{5.8}$$

which coincides, to order 1 in *e*, with the expression given in (2.19b) for the source field. This clearly shows that, in the derivation of (2.19b), we have implicitly calculated the susceptibility of the field. Rather than using this intermediate result, we have preferred in sections 4 and 5 to keep general expressions such as (4.13), (4.14), (4.15), (4.16), which have a clear physical meaning, and to specify the values of C and χ for the field only in these final expressions.

(ii) The free field commutator of (5.3) is a *c*-number ([a, a^+] = 1), proportional to $\hbar$ [see expression (2.3) of $\mathcal{A}_k$]. It follows that the linear susceptibility $\chi^{(R)}$ of the field is independent of the state of the field, and independent of $\hbar$. Therefore the classical and quantum linear susceptibilities coincide. Since the source field is directly related to $\chi^{(R)}$ (see previous remark), it has the same expression in both classical and quantum theories, and this explains why self reaction forces are formally identical in classical and quantum Abraham-Lorentz equations.

We consider now the atomic statistical functions. Their calculation is also straightforward. Using (5.1) in (4.9) and (4.10), replacing $p_{if}(t)$ by $e^{iH_a t/\hbar} p_i e^{-iH_a t/\hbar}$ and introducing some closure relations, we get :

$$C_{ij}^{(S,a)}(\tau) = \frac{1}{2} \frac{e^2}{m^2} \sum_b \{ \langle a | p_i | b \rangle \langle b | p_j | a \rangle e^{i\omega_{ab}\tau} + \langle a | p_j | b \rangle \langle b | p_i | a \rangle e^{-i\omega_{ab}\tau} \} \tag{5.9}$$

$$\chi_{ij}^{(S,a)}(\tau) = \frac{i}{\hbar} \frac{e^2}{m^2} \theta(\tau) \sum_b \{ \langle a | p_i | b \rangle \langle b | p_j | a \rangle e^{i\omega_{ab}\tau} - \langle a | p_j | b \rangle \langle b | p_i | a \rangle e^{-i\omega_{ab}\tau} \} \tag{5.10}$$

where $\hbar\omega_{ab} = E_a - E_b$.

The Fourier transforms of (5.9) and (5.10) are :

$$\hat{C}_{ij}^{(S,a)}(\omega) = \frac{1}{2} \frac{e^2}{m^2} \sum_b \{ \langle a | p_i | b \rangle \langle b | p_j | a \rangle \delta(\omega - \omega_{ab}) + \langle a | p_j | b \rangle \langle b | p_i | a \rangle \delta(\omega + \omega_{ab}) \} \tag{5.11}$$

$$\hat{\chi}_{ij}^{(S,a)}(\omega) = -\frac{e^2}{2 \pi \hbar m^2} \sum_b \left\{ \langle a | p_j | b \rangle \langle b | p_i | a \rangle \mathcal{S}\left(\frac{1}{\omega + \omega_{ab}} \right) - \langle a | p_i | b \rangle \langle b | p_j | a \rangle \mathcal{S}\left(\frac{1}{\omega - \omega_{ab}} \right) \right\}$$

$$- i \frac{e^2}{2 \hbar m^2} \sum_b \{ \langle a | p_j | b \rangle \langle b | p_i | a \rangle \delta(\omega + \omega_{ab}) - \langle a | p_i | b \rangle \langle b | p_j | a \rangle \delta(\omega - \omega_{ab}) \} \tag{5.12}$$

where $\mathcal{S}$ means principal part.

The first line of (5.12), which contains only principal parts, is the reactive part χ' of the susceptibility, whereas the second line, which contains only δ-functions, is the dissipative part $i\chi''$ [26-27].

5.2 RADIATIVE CORRECTIONS FOR AN ATOMIC ELECTRON. — 5.2.1 *Calculations of* $(\delta E_a)_{vf}$ *and* $(\delta E_a)_{sr}$. — We can now use the results of the previous section (5.1) for evaluating the two integrals appearing in the expressions (4.13) and (4.15) giving the energy shifts of the atomic level *a* respectively due to vacuum fluctuations and self reaction. We must not forget to add $\delta m_2 c^2$ to $(\delta E_a)_{vf}$ and $\delta m_1 c^2$ to $(\delta E_a)_{sr}$ where $\delta m_2 c^2$ and $\delta m_1 c^2$

are given by (3.5) and (2.12) and represent overall energy shifts respectively due to vacuum fluctuations and self reaction (see § 3.1).

Using (4.15), (5.5) and (5.9), we first calculate :

$$-\frac{1}{2} \sum_{ij} \int_{-\infty}^{+\infty} d\tau \, \chi_{ij}^{(R)}(\tau) \, C_{ij}^{(S,a)}(\tau) = -\frac{\omega_M}{6\,\pi^2\,\varepsilon_0\,c^3} \sum_{ij} \delta_{ij} \, C_{ij}^{(S,a)}(0)$$

$$= -\frac{\omega_M}{6\,\pi^2\,\varepsilon_0\,c^3} \frac{e^2}{m^2} \langle a \,|\, \mathbf{p}^2 \,|\, a \rangle \tag{5.13}$$

which gives :

$$(\delta E_a)_{\rm sr} = \delta m_1 \, c^2 - \frac{4}{3} \frac{\delta m_1}{m} \left\langle a \left| \frac{\mathbf{p}^2}{2\,m} \right| a \right\rangle. \tag{5.14}$$

For $(\delta E_a)_{\rm vf}$, we first use the Parseval-Plancherel identity

$$-\frac{1}{2} \int_{-\infty}^{+\infty} d\tau \, C_{ij}^{(R)}(\tau) \, \chi_{ij}^{(S,a)}(\tau) = -\pi \int_{-\infty}^{+\infty} d\omega \, \hat{C}_{ij}^{(R)*}(\omega) \, \hat{\chi}_{ij}^{(S,a)}(\omega). \tag{5.15}$$

The integral over ω is then performed. Using (5.6) and (5.12), we get for (5.15) :

$$-\frac{1}{2} \int_{-\infty}^{+\infty} d\tau \, C_{ij}^{(R)}(\tau) \, \chi_{ij}^{(S,a)}(\tau) = \frac{e^2}{24\,\pi^2\,\varepsilon_0\,m^2\,c^3} \sum_b |\langle a \,|\, \mathbf{p} \,|\, b \rangle|^2 \int_{-\omega_M}^{\omega_M} d\omega \, |\omega| \left\{ \mathcal{P}\!\left(\frac{1}{\omega + \omega_{ab}}\right) - \mathcal{P}\!\left(\frac{1}{\omega - \omega_{ab}}\right) \right\}$$

$$= -\frac{e^2}{6\,\pi^2\,\varepsilon_0\,m^2\,c^3} \sum_b \omega_{ab} \, |\langle a \,|\, \mathbf{p} \,|\, b \rangle|^2 \, \mathrm{Log} \frac{\omega_M}{|\omega_{ab}|}. \tag{5.16}$$

(Terms in $1/\omega_M$ have been neglected in (5.16).)

As in similar calculations [29], we introduce an average atomic frequency $\bar{\omega}$ defined by :

$$\sum_b \omega_{ab} \, |\langle a \,|\, \mathbf{p} \,|\, b \rangle|^2 \, \mathrm{Log} \frac{\omega_M}{|\omega_{ab}|} = \sum_b \omega_{ab} \, |\langle a \,|\, \mathbf{p} \,|\, b \rangle|^2 \, \mathrm{Log} \frac{\omega_M}{\bar{\omega}}. \tag{5.17}$$

The summation over b in (5.17) can then be easily done :

$$\sum_b \omega_{ab} \, |\langle a \,|\, \mathbf{p} \,|\, b \rangle|^2 = \frac{1}{2\hbar} \langle a \,|\, [[H_s, \mathbf{p}], \mathbf{p}] \,|\, a \rangle$$

$$= -\frac{\hbar}{2} \langle a \,|\, \Delta V_0(\mathbf{r}) \,|\, a \rangle. \tag{5.18}$$

Finally, one gets for $(\delta E_a)_{\rm vf}$:

$$(\delta E_a)_{\rm vf} = \frac{e^2 \, \hbar}{12\,\pi^2\,\varepsilon_0\,m^2\,c^3} \, \mathrm{Log} \frac{\omega_M}{\bar{\omega}} \langle a \,|\, \Delta V_0(\mathbf{r}) \,|\, a \rangle + \delta m_2 \, c^2. \tag{5.19}$$

It is important to note that, in the derivation of (5.14) and (5.19), we have not used approximations such as the two-level approximation, or the rotating wave approximation. The energy level shifts are due to virtual transitions involving non resonant couplings. Consequently, a correct derivation of these shifts must take into account all atomic states and both positive and negative frequency components of the field.

5.2.2. *Main effect of self reaction : modification of kinetic energy due to a mass renormalization.* — The

first term of (5.14) has already been interpreted as the increase of the rest mass energy of the electron due to its Coulomb field. The last term can be considered as the first order correction to the average kinetic energy of the electron when m is replaced by $m + 4\,\delta m_1/3$:

$$\left\langle a \left| \frac{\mathbf{p}^2}{2(m + \frac{4}{3}\delta m_1)} \right| a \right\rangle = \left\langle a \left| \frac{\mathbf{p}^2}{2\,m} \right| a \right\rangle \times$$

$$\times \left(1 - \frac{4}{3} \frac{\delta m_1}{m} + \cdots \right). \tag{5.20}$$

The electron is surrounded by its Coulomb field, and when one pushes the electron, one has also to push its Coulomb field (electromagnetic mass).

The mass corrections appearing in the two terms of (5.14) are not the same. This discrepancy is due to the non covariant cut-off (see discussion of § 2.1.4), and also exists in classical theory.

Finally, it must be noted that, since the $2s_{1/2}$ and $2p_{1/2}$ states of hydrogen have the same average kinetic energy, a mass correction produces equal shifts for the two levels and cannot remove their degeneracy. Self reaction alone cannot therefore explain the Lamb-shift.

5.2.3 Main effect of vacuum fluctuations : modification of potential energy.

The first term of (5.19) coincides with the standard non relativistic expression for the Lamb-shift [29]. It appears as a correction to the potential $V_0(\mathbf{r})$ which becomes $V_0(\mathbf{r}) + \delta V_0(\mathbf{r})$ where

$$\delta V_0(\mathbf{r}) = \frac{e^2 \hbar}{12 \pi^2 \varepsilon_0 m^2 c^3} \operatorname{Log} \frac{\omega_M}{\bar{\omega}} \Delta V_0(\mathbf{r}) . \qquad (5.21)$$

If $V_0(\mathbf{r})$ is the Coulomb potential of a nucleus located at the origin, $\Delta V_0(\mathbf{r})$ is proportional to $\delta(\mathbf{r})$, and therefore only s states are shifted by such a correction, which explains in particular how the degeneracy between $2s_{1/2}$ and $2p_{1/2}$ can be removed.

Welton has pointed out [3] that a correction of the same type as (5.21) would be obtained, if the electron was submitted to a fluctuating classical field, with frequencies large compared to the atomic frequencies. The electron, vibrating in such a fluctuating field, averages the external static potential over a finite volume. If the spectral density of this random perturbation is identified with the one of vacuum fluctuations, one gets for the coefficient of $\Delta V_0(\mathbf{r})$ the same value as in (5.21), $\bar{\omega}$ being simply replaced by a low frequency cut-off. Welton's analysis establishes a connection between Lamb-shifts of atomic levels and vacuum fluctuations and provides a clear and simple physical picture.

Our choice of the symmetric order in (3.9) ascribes corrections such as (5.21) to vacuum fluctuations and entirely legitimates Welton's interpretation for the Lamb-shift.

We have already seen (§ 3.1.ii) that vacuum fluctuations are also responsible for a correction δm_2 to the electron mass (last term of (5.19)).

Remarks : (i) It may appear surprising that our calculation doesn't give any correction to the kinetic energy associated with the mass correction δm_2 due to vacuum fluctuations. One would expect to find, as in the previous section, a term of the order of

$$- \frac{\delta m_2}{m} \left\langle a \left| \frac{\mathbf{p}^2}{2 m} \right| a \right\rangle . \qquad (5.22)$$

Actually, coming back to the expressions (2.12) and (3.5) of δm_1 and δm_2, and introducing the fine structure constant $\alpha = e^2/4 \pi \varepsilon_0 \hbar c$, one can write

$$\begin{cases} \dfrac{\delta m_1}{m} = \dfrac{\alpha}{\pi} \dfrac{\hbar \omega_M}{mc^2} & (5.23a) \\[2mm] \dfrac{\delta m_2}{m} = \dfrac{\alpha}{2 \pi} \left(\dfrac{\hbar \omega_M}{mc^2} \right)^2 . & (5.23b) \end{cases}$$

Therefore, it clearly appears that $\delta m_2/m$ is of higher order in $1/c$ than $\delta m_1/m$. This explains why the correction to the kinetic energy associated with δm_2 is not given by our calculation which is limited to the lowest order in $1/c$. The basic Hamiltonian (2.13) does not contain any relativistic correction. A more precise calculation including in the Hamiltonian relativistic corrections up to order $1/c^2$ [4] (and using an effective Hamiltonian method for evaluating radiative corrections) actually gives the expected correction (5.22).

(ii) The present calculation (as well as the one of reference [4]) does not include of course any multi-particle effect (virtual pair creation). It is well known [19] that many particle effects reduce the divergence of the electron self-energy $(\delta m_1 + \delta m_2) c^2$, with respect to the cut-off ω_M. Instead of having a linear and quadratic divergence (see (5.23)), one gets a logarithmic one. Also, new correction terms, associated with vacuum polarization effects, appear.

5.2.4 Classical versus quantum effects.

A striking difference can be pointed out between the contributions of self reaction and vacuum fluctuations to radiative corrections : $\hbar$ does not appear in $(\delta E_a)_{sr}$ [see (5.14) and the expression (2.12) of δm_1], whereas $\hbar$ does appear in both terms of $(\delta E_a)_{vf}$ [see (5.19) and the expression (3.5) of δm_2].

The fact that self reaction corrections are purely classical (independent of $\hbar$) is not surprising. We have already explained (see remark (ii) of section 5.1) why self reaction terms are identical in both classical and quantum theories.

On the other hand, vacuum fluctuation corrections have an essentially quantum nature since they are due to the non zero mean square value of the fields in the vacuum, which is a pure quantum effect. It must be noted however that, once the correlation function of vacuum fluctuations is computed from the quantum theory of radiation, their effect on the atom (to the lowest order in α) may be evaluated semi-classically, since we have shown in § 4.3 that reservoir fluctuations have the same effect (to the lowest order) as a classical random field having same correlation function. This explains why pure quantum effects, such as those produced by the vacuum fluctuations of the quantized radiation field, can be calculated as if a classical random field, with a power spectral density equal to $\hbar \omega/2$ per mode, was acting upon the atom [30].

To summarize, our choice of the symmetric order in (3.9) leads to self reaction corrections which are strictly equivalent to the corresponding classical ones, whereas vacuum fluctuations appear to be responsible for pure quantum effects which can be however computed semi-classically, once the correlation function of vacuum fluctuations is given.

5.2.5 Spin and magnetic effects. Interpretation of the spin anomaly $g - 2$.

— In this section, we take into account the spin S of the electron and the corresponding magnetic moment

$$\mathbf{M_s} = \frac{e}{m}\mathbf{S}. \tag{5.24}$$

Even in the absence of any external static magnetic field $\mathbf{B_0}$, $\mathbf{M_s}$ interacts with the magnetic field $\mathbf{B}$ of the transverse radiation field. We should add to the interaction Hamiltonian V given in (3.3) a term.

$$- \mathbf{M_s} \cdot \mathbf{B}(0) = - \frac{e}{m}\mathbf{S} \cdot \mathbf{B}(0) \tag{5.25}$$

describing such a coupling. This would introduce in the final expressions of radiative corrections new correlation functions and new linear susceptibilities involving two components of $\mathbf{B}$, or one component of $\mathbf{B}$ and one component of $\mathbf{A}$. Since an extra $1/c$ factor appears in the expansion of $\mathbf{B}$ in plane waves [see expression (2.3)], we conclude that the new radiative corrections associated with (5.25) would be at least one order in $1/c$ higher than those calculated previously, and which, according to (5.13) and (5.19) are in e^2/c^3 (or α/mc^2). If we restrict our calculations to the lowest order in $1/c$, as we do in the non relativistic approach used in this paper, we can therefore ignore the magnetic couplings of the spin with the radiation field and neglect (5.25) ([3]).

The same argument does not apply of course to the interaction of S with an external static magnetic field $\mathbf{B_0}$ deriving from the static vector potential $\mathbf{A_0}$:

$$\mathbf{B_0}(\mathbf{R}) = \nabla \times \mathbf{A_0}(\mathbf{R}). \tag{5.26}$$

We must add to the atomic Hamiltonian H_s a new term describing the interaction of $\mathbf{M_s}$ with the static magnetic field $\mathbf{B_0}$ at the electron position

$$- \mathbf{M_s} \cdot \mathbf{B_0}(\mathbf{r}) = - \frac{e}{m}\mathbf{S} \cdot \mathbf{B_0}(\mathbf{r}). \tag{5.27}$$

We must also replace the electron momentum $\mathbf{p}$ by :

$$\pi_0 = \mathbf{p} - e\mathbf{A_0}(\mathbf{r}). \tag{5.28}$$

To summarize, if, at the lowest order in $1/c$, i.e. at

order e^2/c^3, we want to include spin and magnetic effects, we must use :

$$H_s = \frac{\pi_0^2}{2m} + V_0(\mathbf{r}) - \frac{e}{m}\mathbf{S} \cdot \mathbf{B_0}(\mathbf{r}) \tag{5.29}$$

instead of (3.1), and replace $\mathbf{p}$ by π_0 in the first term of (3.3) :

$$- \frac{e}{m}\mathbf{p} \cdot \mathbf{A}(0) \rightarrow - \frac{e}{m}\pi_0 \cdot \mathbf{A}(0). \tag{5.30}$$

What are the corresponding changes in $(\delta E_a)_{sr}$ and $(\delta E_a)_{vf}$? Since the field operators remain unchanged in (5.30), we still use (5.4) and (5.5) for $C^{(R)}$ and $\chi^{(R)}$. On the other hand, we must change $\mathbf{p}$ into π_0 in the expressions (5.9) and (5.10) of $C^{(S)}$ and $\chi^{(S)}$.

Consider first the modifications occurring for $(\delta E_a)_{sr}$. The only change in (5.14) is that we have $\pi_0^2/2m$ instead of $\mathbf{p}^2/2m$. Since $\pi_0^2/2m$ has the physical meaning of a kinetic energy in presence of the static vector potential $\mathbf{A_0}$, we conclude that the main effect of self reaction is, as before, to change the mass appearing in the kinetic energy

$$\frac{\pi_0^2}{2m} \rightarrow \frac{\pi_0^2}{2(m + \frac{4}{3}\delta m_1)}. \tag{5.31}$$

It must be emphasized that, at this order in $1/c$, the mass renormalization due to self reaction does not affect the last term of (5.29). The mass m which appears in the spin magnetic moment $e\mathbf{S}/m$ remains unchanged. We don't get any term of the form

$$+ \frac{\delta m_1}{m}\frac{e}{m}\mathbf{S} \cdot \mathbf{B_0}. \tag{5.32}$$

We will come back later on this important point, when discussing the origin of the spin anomaly $g-2$.

We now discuss the modifications for $(\delta E_a)_{vf}$. The calculations are very similar to those of § 5.2.1, the only difference being that, in the double commutator of (5.18), we must use the new expression (5.29) of H_s and replace $\mathbf{p}$ by π_0. We have therefore to calculate :

$$\frac{1}{2\hbar}\left\langle a \left| \left[\left[\frac{\pi_0^2}{2m} + V_0(\mathbf{r}) - \frac{e}{m}\mathbf{S} \cdot \mathbf{B_0}(\mathbf{r}), \pi_0 \right], \pi_0 \right] \right| a \right\rangle \tag{5.33}$$

If we suppose that $\mathbf{B_0}(\mathbf{r})$ is homogeneous (independent of $\mathbf{r}$) and if we keep only terms linear in $\mathbf{B_0}$, expression (5.33) reduces to (5.18). Thus, for homogeneous weak static magnetic fields, vacuum fluctuations do not introduce any new radiative correction related to spin and magnetic effects.

We have now at our disposal all what is needed for discussing the contribution of self reaction and vacuum fluctuations to the electron dynamics in presence of a weak homogeneous static magnetic

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([3]) If we would like to go to higher orders in $1/c$, we should include relativistic corrections in the Hamiltonian and retardation effects.

field. Combining the previous results, the corrected atomic Hamiltonian (including radiative corrections) can be written :

$$\frac{\pi_0^2}{2(m + \frac{4}{3}\delta m_1)} + V_0(\mathbf{r}) + \delta V_0(\mathbf{r}) - \frac{e}{m}\mathbf{S}.\mathbf{B}_0(\mathbf{r})$$

$$(5.34)$$

where corrections including δm_1 are due to self reaction and $\delta V_0(\mathbf{r})$ to vacuum fluctuations.

The spin magnetic moment appearing in the last term of (5.34) can be written as :

$$\mathbf{M}_s = \frac{e}{m}\mathbf{S} = 2\frac{e}{2m}\mathbf{S}. \qquad (5.35)$$

In terms of the « bare » (uncorrected) mass, the g factor of the electron spin is 2. But, the mass which is measured experimentally, in deflection experiments, is the renormalized mass, i.e. the mass which appears in the corrected kinetic energy

$$\overline{m} = m + \frac{4}{3}\delta m_1 \qquad (5.36)$$

so that, if we reexpress $\mathbf{M}_s$ in terms of $\overline{m}$, we have from (5.35)

$$\mathbf{M}_s = \frac{e}{m}\mathbf{S} = g\frac{e}{2\overline{m}}\mathbf{S} \qquad (5.37)$$

with

$$g = 2\frac{\overline{m}}{m} = 2\left(1 + \frac{4}{3}\frac{\delta m_1}{m}\right) > 2. \qquad (5.38)$$

So, it clearly appears that the positive sign of $g-2$ is due to the fact that self reaction corrects only to lowest order the kinetic energy and not the magnetic coupling between S and $\mathbf{B}_0$. The motion of the charge is slowed down but not the precession of the spin. This is easy to understand. In the non relativistic limit we are considering in this paper, electric effects predominate over magnetic ones and self reaction is stronger for a charge than for a magnetic moment. We therefore arrive at the same conclusions as other treatments [4, 31].

If the calculation was pushed to higher orders in $1/c$, as in [4], we would get corrections to the spin magnetic moment, especially those due to the vacuum

fluctuations of the magnetic field $\mathbf{B}(0)$ of the radiation field which exert a fluctuating torque on $\mathbf{M}_s$, producing an angular vibration of the spin and, consequently, a decrease of the effective magnetic moment. This is the equivalent of Welton's picture for $g - 2$ which would produce a negative spin anomaly if this was the only mechanism. We understand now the failure of such a picture. For $g - 2$, the predominant physical mechanism is self reaction which slows down the motion of the electric charge.

5.3 RATE OF EXCHANGE OF ENERGY BETWEEN THE ELECTRON AND THE RADIATION FIELD. — 5.3.1 *Contribution of self reaction.* — We start from (4.16) and we use the expressions (5.7) of χ^R and (5.11) of C^S. Because of the δ function appearing in (5.11), the integral over ω is readily done, and we get for the rate of energy loss due to self reaction by the electron in state a

$$\left\langle 0, a \left| \left(\frac{d}{dt} H_s\right)_{sr} \right| 0, a \right\rangle =$$

$$= -\frac{e^2}{6\pi m^2 \varepsilon_0 c^3} \sum_b \sum_i |\langle a | p_i | b \rangle|^2 \omega_{ab}^2. \quad (5.39)$$

Now, we write [4]

$$\frac{1}{m}\langle a | p_i | b \rangle \omega_{ab} = \frac{1}{m\hbar}\langle a | [H_s, p_i] | b \rangle$$

$$= -\frac{i}{m}\langle a | \dot{p}_i | b \rangle$$

$$= -i\langle a | \ddot{r}_i | b \rangle. \qquad (5.40)$$

Finally, by using (5.40) and the closure relation over b, we transform (5.39) into :

$$\left\langle 0, a \left| \left(\frac{d}{dt} H_s\right)_{sr} \right| 0, a \right\rangle = -\frac{2}{3}\frac{e^2}{4\pi\varepsilon_0 c^3}\langle a | (\ddot{r})^2 | a \rangle.$$

$$(5.41)$$

Such a result is extremely simple and exactly coincides with what is found in classical radiation theory. The rate of radiation of electromagnetic energy is proportional to the square of the acceleration of the radiating charge, the proportionality coefficient being just the one appearing in (5.41). We note also that, if self reaction was alone, the atomic ground state would not be stable, since the square of the acceleration has a non zero average value in such a state.

5.3.2 *Contribution of vacuum fluctuations.* — We now use (4.14) and the expressions (5.6) of $C^{(R)}$ and (5.12) of $\chi^{(S)}$. This gives

$$\left\langle 0, a \left| \left(\frac{d}{dt} H_s\right)_{vf} \right| 0, a \right\rangle = \frac{e^2}{12\pi\varepsilon_0 m^2 c^3} \int_{-\infty}^{+\infty} d\omega\, \omega\, |\omega| \sum_b \sum_i |\langle a | p_i | b \rangle|^2 (\delta(\omega + \omega_{ab}) - \delta(\omega - \omega_{ab}))$$

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[4] The atomic operators appearing in $\chi^{(S)}$ are free atomic operators. This is why their time derivative is given by the commutator with H_s (and not with the total Hamiltonian H).

$$= - \frac{e^2}{6\,\pi\varepsilon_0\,m^2\,c^3} \sum_b \sum_i |\langle a|p_i|b\rangle|^2\,\omega_{ab}\,|\omega_{ab}|. \tag{5.42}$$

Using (5.40), and distinguishing the terms $\omega_{ab} > 0$ $(E_a > E_b)$ and the terms $\omega_{ab} < 0$ $(E_a < E_b)$, we get :

$$\left\langle 0,a \left| \left(\frac{dH_s}{dt}\right)_{vf} \right| 0,a \right\rangle = \frac{2}{3}\frac{e^2}{4\,\pi\varepsilon_0\,c^3} \left\{ \sum_{\substack{b \\ E_b > E_a}} \langle a|\ddot{\mathbf{r}}|b\rangle.\langle b|\ddot{\mathbf{r}}|a\rangle - \sum_{\substack{b \\ E_b < E_a}} \langle a|\ddot{\mathbf{r}}|b\rangle.\langle b|\ddot{\mathbf{r}}|a\rangle \right\}. \tag{5.43}$$

The first line describes an absorption of energy by the electron which jumps from a to a higher state b, whereas the second line describes an emission of energy by jumps to lower states. This is in agreement with the picture of a random field inducing in the atomic system both downwards and upwards transitions.

Now, coming back to (5.41), we can reintroduce the closure relation over b between $\ddot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$, which gives :

$$\left\langle 0,a \left| \left(\frac{d}{dt}H_s\right)_{sr} \right| 0,a \right\rangle = -\frac{2}{3}\frac{e^2}{4\,\pi\varepsilon_0\,c^3} \sum_b \langle a|\ddot{\mathbf{r}}|b\rangle.\langle b|\ddot{\mathbf{r}}|a\rangle$$

$$= -\frac{2}{3}\frac{e^2}{4\,\pi\varepsilon_0\,c^3} \left\{ \sum_{\substack{b \\ E_b > E_a}} \langle a|\ddot{\mathbf{r}}|b\rangle.\langle b|\ddot{\mathbf{r}}|a\rangle + \sum_{\substack{b \\ E_b < E_a}} \langle a|\ddot{\mathbf{r}}|b\rangle.\langle b|\ddot{\mathbf{r}}|a\rangle \right\}. \tag{5.44}$$

Adding (5.43) and (5.44), we get for the total rate of energy loss by the electron in state a

$$\left\langle 0,a \left| \frac{d}{dt}H_s \right| 0,a \right\rangle = -\frac{4}{3}\frac{e^2}{4\,\pi\varepsilon_0\,c^3} \sum_{\substack{b \\ E_b < E_a}} \langle a|\ddot{\mathbf{r}}|b\rangle.\langle b|\ddot{\mathbf{r}}|a\rangle. \tag{5.45}$$

This satisfactory result means that the electron in the vacuum can only loose energy by cascading downwards to lower energy levels. In particular, the ground state is stable since it is the lowest state.

The previous discussion clearly shows that the ground state cannot be stable in absence of vacuum fluctuations which exactly balance the energy loss due to self reaction [28]. In other words, if self reaction was alone, the ground state would collapse and the atomic commutation relation $[x, p] = i\hbar$ would not hold. Such a collapse is prevented by vacuum fluctuations which actually originate from the quantum nature of the field, i.e. from the commutation relation $[a, a^+] = 1$. We have here an illustration of a very general principle of quantum mechanics. When two isolated systems interact (here the atom and the field), treating one of them quantum mechanically and the other one semi-classically leads to inconsistencies [32]. The field commutation relations are necessary for preserving the atomic ones and vice versa.

Such a procedure is very general and can be extended to the case of a small system S interacting with a large reservoir $\mathfrak{R}$. The results of the calculation can be expressed ([5]) in terms of simple statistical functions of the two interacting systems, leading to simple physical pictures : $\mathfrak{R}$ fluctuates and polarizes S (reservoir fluctuations effects); S fluctuates and polarizes $\mathfrak{R}$ (self reaction effects).

When applied to the case of an atomic electron interacting with the vacuum field, such a procedure gives results in complete agreement with the usual pictures associated with vacuum fluctuations and self reaction. All self reaction effects, which are independent of $\hbar$, are strictly identical to those derived from classical radiation theory. All vacuum fluctuation effects, which are proportional to $\hbar$, can be interpreted by considering the vibration of the electron induced by a random field having a spectral power density equal to $\hbar\omega/2$ per mode.

6. **Conclusion.** — We have removed the apparent indetermination in the separation of vacuum fluctuations and self reaction by imposing to the corresponding rates of variation to have a well-defined physical meaning (hermiticity requirements).

([5]) It must be kept in mind that all the calculations have been limited to order 2 in the coupling constant. At higher orders, cross terms would appear between reservoir fluctuations and self reaction.

Appendix A : Calculation of the source fields $A_s(0, t)$ and $E_{\perp s}(0, t)$. — Equations (2.2a) and (2.2b) give the expressions of A and $E_\perp$ in terms of the creation and annihilation operators :

$$A(0, t) = \sum_{\substack{k\varepsilon \\ |k| < k_M}} \mathcal{A}_k \, \varepsilon a_{k\varepsilon}(t) + hc \tag{A.1a}$$

$$E_\perp(0, t) = \sum_{\substack{k\varepsilon \\ |k| < k_M}} \mathcal{E}_k \, \varepsilon a_{k\varepsilon}(t) + hc . \tag{A.1b}$$

Inserting (2.17) into these two equations, one gets the expression of $A_s(0, t)$ and $E_{\perp s}(0, t)$:

$$A_s(0, t) = \sum_{\substack{k\varepsilon \\ |k| < k_M}} i \frac{e\mathcal{A}_k^2}{m\hbar} \int_{t_0}^t dt' \, e^{-i\omega(t-t')} \, \varepsilon(\varepsilon^*.\pi(t')) + hc \tag{A.2a}$$

$$E_{\perp s}(0, t) = \sum_{\substack{k\varepsilon \\ |k| < k_M}} i \frac{e\mathcal{A}_k \mathcal{E}_k}{m\hbar} \int_{t_0}^t dt' \, e^{-i\omega(t-t')} \, \varepsilon(\varepsilon^*.\pi(t')) + hc . \tag{A.2b}$$

We now permute the summation over k, ε and the integration on t', the angular summation is easily performed and we get :

$$A_s(0, t) = - \frac{e}{3 \pi\varepsilon_0 c^3 m} \int_{t_0}^t dt' \, \pi(t') \, \delta_M'(t - t') \tag{A.3a}$$

$$E_{\perp s}(0, t) = \frac{e}{3 \pi\varepsilon_0 c^3 m} \int_{t_0}^t dt' \, \pi(t') \, \delta_M''(t - t') \tag{A.3b}$$

where the function $\delta_M(\tau)$ is given by

$$\delta_M(\tau) = \frac{1}{2 \pi} \int_{-\omega_M}^{+\omega_M} d\omega \, e^{-i\omega\tau} . \tag{A.4}$$

This function $\delta_M(\tau)$ is symmetric, centred on $\tau = 0$, has a width equal to $1/\omega_M$ and satisfies the equation

$$\int_{-\infty}^{+\infty} d\tau \, \delta_M(\tau) = 1 . \tag{A.5}$$

Equations (A.3a) and (A.3b) can be written, by putting $\tau = t - t'$ and taking t_0 equal to $-\infty$:

$$A_s(0, t) = - \frac{e}{3 \pi\varepsilon_0 c^3 m} \int_0^{+\infty} d\tau \, \pi(t - \tau) \, \delta_M'(\tau) \tag{A.6a}$$

$$E_{\perp s}(0, t) = \frac{e}{3 \pi\varepsilon_0 c^3 m} \int_0^{+\infty} d\tau \, \pi(t - \tau) \, \delta_M''(\tau) . \tag{A.6b}$$

Using an integration by parts, one gets

$$A_s(0, t) = \frac{e}{3 \pi\varepsilon_0 c^3 m} \delta_M(0) \, \pi(t) - \frac{e}{3 \pi\varepsilon_0 c^3 m} \int_0^{+\infty} d\tau \, \dot{\pi}(t - \tau) \, \delta_M(\tau) \tag{A.7a}$$

$$E_{\perp s}(0, t) = - \frac{e}{3 \pi\varepsilon_0 c^3 m} \delta_M(0) \, \dot{\pi}(t) + \frac{e}{3 \pi\varepsilon_0 c^3 m} \int_0^{+\infty} d\tau \, \ddot{\pi}(t - \tau) \, \delta_M(\tau) . \tag{A.7b}$$

The characteristic times for the evolution of $\pi(t)$ are very long compared to the width $1/\omega_M$ of $\delta_M(\tau)$ [see Eq. (2.11)]. We can therefore replace in (A.7a) and (A.7b) $\dot{\pi}(t - \tau)$ and $\ddot{\pi}(t - \tau)$ by $\dot{\pi}(t)$ and $\ddot{\pi}(t)$. The remaining integral of $\delta_M(\tau)$ from $\tau = 0$ to $\tau = \infty$ is equal to $1/2$, as a consequence of the symmetry of $\delta_M(\tau)$. One finally gets

$$A_s(0, t) = \frac{e\omega_M}{3 \pi^2 \varepsilon_0 c^3 m} \pi(t) - \frac{e}{6 \pi \varepsilon_0 c^3 m} \dot{\pi}(t) \tag{A.8a}$$

$$E_{\perp s}(0, t) = - \frac{e\omega_M}{3 \pi^2 \varepsilon_0 c^3 m} \dot{\pi}(t) + \frac{e}{6 \pi \varepsilon_0 c^3 m} \ddot{\pi}(t) \tag{A.8b}$$

(A.8a) and (A.8b) are nothing but (2.19b) and (2.21b) using the expression of δm_1 given in equation (2.12).

Appendix B : Correlation function and linear susceptibility of the field. — The correlation function of the field is given [cf. Eq. (5.2)] :

$$C_{ij}^{(R)}(\tau) = \frac{1}{2} \langle 0 | A_{if}(0, t) A_{jf}(0, t - \tau) + A_{jf}(0, t - \tau) A_{if}(0, t) | 0 \rangle \tag{B.1}$$

where the operator $A_f(0, t)$ is the free vector potential. Using its expansion in plane waves, one gets

$$C_{ij}^{(R)}(\tau) = \frac{1}{2} \sum_{k\varepsilon} \mathcal{A}_k^2 \varepsilon_i \varepsilon_j \langle 0 | a_{k\varepsilon f}(t) a_{k\varepsilon f}^+(t - \tau) + a_{k\varepsilon f}(t - \tau) a_{k\varepsilon f}^+(t) | 0 \rangle$$

$$= \frac{1}{2} \sum_{k\varepsilon} \mathcal{A}_k^2 \varepsilon_i \varepsilon_j (e^{-i\omega\tau} + e^{i\omega\tau}). \tag{B.2}$$

Replacing the sum by an integral and using the expression (2.3) of $\mathcal{A}_k$, one gets

$$C_{ij}^{(R)}(\tau) = \delta_{ij} \frac{\hbar}{12 \pi^2 \varepsilon_0 c^3} \int_0^{\omega_M} d\omega\, \omega (e^{i\omega\tau} + e^{-i\omega\tau}). \tag{B.3}$$

This can also be written :

$$C_{ij}^{(R)}(\tau) = \delta_{ij} \frac{\hbar}{12 \pi^2 \varepsilon_0 c^3} \int_{-\omega_M}^{\omega_M} d\omega\, |\omega| e^{i\omega\tau}. \tag{B.4}$$

The linear susceptibility is calculated in the same way. Starting from

$$\chi_{ij}^{(R)}(\tau) = \frac{i}{\hbar} \langle 0 | [A_{if}(0, t), A_{jf}(0, t - \tau)] | 0 \rangle \theta(\tau) \tag{B.5}$$

one gets

$$\chi_{ij}^{(R)}(\tau) = \frac{i}{\hbar} \delta_{ij} \frac{\hbar}{6 \pi^2 \varepsilon_0 c^3} \int_0^{\omega_M} d\omega\, \omega (e^{-i\omega\tau} - e^{+i\omega\tau}) \theta(\tau)$$

$$= - \frac{\delta_{ij}}{3 \pi \varepsilon_0 c^3} \delta_M'(\tau) \theta(\tau). \tag{B.6}$$

In this paper, the susceptibility of the field always appears in expressions such as

$$\int_{-\infty}^{+\infty} \chi_{ij}^{(R)}(\tau) f_{ij}^{(s)}(\tau) d\tau \tag{B.7}$$

where $f_{ij}^{(s)}(\tau)$ is a function concerning the small system S. The characteristic times of evolution of $f_{ij}^{(s)}$ are then much larger than $1/\omega_M$ so that one can proceed in the same way as for (A.6a). Using an integration by parts, one finds that

$$\chi_{ij}^{(R)}(\tau) = \frac{\delta_{ij}}{3 \pi^2 \varepsilon_0 c^3} \left[\omega_M \delta(\tau) - \frac{\pi}{2} \delta'(\tau) \right] \tag{B.8}$$

where δ here acts on the slowly varying functions $f_{ij}^{(s)}(\tau)$ as a true delta function.

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