Supplemental Material for
Magnetic dipolar interaction between hyperfine clock states in a planar alkali Bose gas

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Restriction of the dipole-dipole interaction to the clock state manifold

In this section, we evaluate the action of the magnetic dipole-dipole interaction inside the two-level manifold relevant for the clock transition. First, using the general expression for the coupling between a spin 1/2 (here the outer electron) and a spin i (here the ^{87}Rb nucleus with $i = 3/2$), we obtain the decomposition of the clock states on the basis $|s_Z, i_Z\rangle$:

$$\begin{aligned}|1\rangle &\equiv |F=1, m_Z=0\rangle \\ &= \frac{1}{\sqrt{2}}\left(-|-\frac{1}{2};+\frac{1}{2}\rangle \;+\; |+\frac{1}{2};-\frac{1}{2}\rangle\right) \qquad (1)\end{aligned}$$

and

$$\begin{aligned}|2\rangle &\equiv |F=2, m=0\rangle \\ &= \frac{1}{\sqrt{2}}\left(|-\frac{1}{2};+\frac{1}{2}\rangle \;+\; |+\frac{1}{2};-\frac{1}{2}\rangle\right). \qquad (2)\end{aligned}$$

The magnetic interaction operator for two electronic spins $\hat{\boldsymbol{s}}_A$ and $\hat{\boldsymbol{s}}_B$ with magnetic moments $\boldsymbol{m}_{A,B} = 2\mu_B \boldsymbol{s}_{A,B}$ is given by

$$\hat{V}_{\rm dd}(r, \boldsymbol{u}) = \frac{\mu_0\mu_B^2}{\pi r^3}[\hat{\boldsymbol{s}}_A \cdot \hat{\boldsymbol{s}}_B - 3(\hat{\boldsymbol{s}}_A\cdot\boldsymbol{u})(\hat{\boldsymbol{s}}_B\cdot\boldsymbol{u})], \qquad (3)$$

where r is the distance between the two dipoles and $\boldsymbol{u}$ is the unit vector connecting them. We calculate the matrix elements of this operator in the basis $\{|11\rangle, |12\rangle, |21\rangle, |22\rangle\}$, restricting to elastic interactions which are the only relevant ones for the experimental time scale. This leaves us with four different matrix elements to compute: V_{1111}, V_{2222}, $V_{1212} = V_{2121}$ and $V_{1221} = V_{2112}$, where $V_{ijkl} = \langle kl|\hat{V}_{\rm dd}|ij\rangle$. The calculation in the basis (1,2) leads to

$$\hat{s}_Z|1\rangle = \frac{1}{2}|2\rangle, \qquad \hat{s}_Z|2\rangle = \frac{1}{2}|1\rangle \qquad (4)$$

The operators $\hat{s}_X$ and $\hat{s}_Y$ couple states with different m_F values and the associated matrix elements V_{ijkl} inside the clock state manifold are zero. The magnetic interaction operator in Eq. (3) thus simplifies to

$$\hat{V}_{\rm dd}(r,\theta) = \frac{\mu_0\mu_B^2}{\pi r^3}\,(1-3\cos^2\theta)\;\hat{s}_{Z,A}\,\hat{s}_{Z,B}. \qquad (5)$$

We deduce that among the four matrix elements mentioned above, only

$$V_{1221} = V_{2112} = \frac{\mu_0\mu_B^2}{4\pi r^3}\,(1-3\cos^2\theta) \qquad (6)$$

is non-zero, where θ is the angle between $\boldsymbol{u}$ and the quantization axis. This shows that MDDI do not modify the interactions between atoms in the same state $|1\rangle$ or $|2\rangle$, but induce a non-local, angle-dependent, exchange interaction. The second-quantized Hamiltonian of the MDDI for the clock transition is thus:

$$\begin{aligned}\hat{H}_{\rm dd}^{(1,2)} &= \frac{\mu_0\mu_B^2}{4\pi}\iint {\rm d}^3r_A\,{\rm d}^3r_B\;\frac{1-3\cos^2\theta}{|\boldsymbol{r}_A-\boldsymbol{r}_B|^3} \\ &\times\;\hat{\Psi}_2^\dagger(\boldsymbol{r}_A)\,\hat{\Psi}_1^\dagger(\boldsymbol{r}_B)\,\hat{\Psi}_2(\boldsymbol{r}_B)\,\hat{\Psi}_1(\boldsymbol{r}_A), \qquad (7)\end{aligned}$$

where the $\hat{\Psi}_i(\boldsymbol{r}_\alpha)$ are the field operators annihilating a particle in state $|i\rangle$ at position $\boldsymbol{r}_\alpha$.

Calculation of δa_{12}

We consider a gas with a density distribution $n(x,y,z) = \rho(x,y)e^{-z^2/\ell_z^2}/(\ell_z\sqrt{\pi})$ subject to a magnetic field $\boldsymbol{B} = B(\cos\Theta\,\boldsymbol{u}_z + \sin\Theta\,\boldsymbol{u}_x)$ which defines the quantization axis Z for the spin states. We compute the mean-field energy associated to magnetic dipole-dipole interactions following Ref. [1]. In Fourier space, we express the mean-field energy as

$$\langle\hat{H}_{\rm dd}\rangle = \frac{1}{2}\frac{1}{(2\pi)^3}\int {\rm d}^3k\;\tilde{n}(\boldsymbol{k})\;\tilde{V}_{\rm dd}(\boldsymbol{k})\;\tilde{n}(-\boldsymbol{k}) \qquad (8)$$

where $\tilde{V}_{\rm dd}(\boldsymbol{k}) = 3\mu_0\mu_B^2[\cos^2\alpha - 1/3]$ is the Fourier transform of the dipole-dipole interaction with α the angle of the wavevector $\boldsymbol{k}$ with respect to $\boldsymbol{B}$. The Fourier transform of the density distribution is given by $\tilde{n}(\boldsymbol{k}) = e^{k_z^2\ell_z^2/4}\tilde{\rho}(k_x,k_y)$, where $\tilde{\rho}(k_x,k_y)$ is the Fourier transform of $\rho(x,y)$. Introducing $k = |\boldsymbol{k}|$, we have $\cos(\alpha) = (k_z\cos\Theta + k_x\sin\Theta)/k$ and we get

$$\langle \hat{H}_{\rm dd} \rangle = \frac{\mu_0 \mu_B^2}{3\sqrt{2\pi}\ell_z} \left\{ \frac{1}{(2\pi)^2} \int {\rm d}^2 k \; \tilde{\rho}(k_x, k_y) \; \tilde{\rho}(-k_x, -k_y) \left[(3\cos^2\Theta - 1) + \left(\frac{k_x^2}{k_\perp^2} \sin^2\Theta - \cos^2\Theta \right) \mathcal{F}(k_\perp \ell_z) \right] \right\}, \quad (9)$$

where we have introduced $k_\perp = \sqrt{k_x^2 + k_y^2}$ and $\mathcal{F}(k_\perp \ell_z) = 3\sqrt{\pi/2} k_\perp \ell_z e^{k_\perp^2 \ell_z^2/2} \,{\rm erfc}(k_\perp \ell_z^2/\sqrt{2})$. For a uniform system, $\rho(k_x, k_y) = \delta(k_x)\delta(k_y)$ is the product of two Dirac delta functions and we recover Eq.(3) of the main text. Consider now the case of spins aligned along the x axis corresponding to $\Theta = \pi/2$, as in Fig. 4 of the main text. For a cloud shape with typical length scales larger than ℓ_z, the influence of the shape of the cloud via the integration over k_x and k_y scales with $\mathcal{F}(k_\perp \ell_z) \sim k_\perp \ell_z$, which is a small parameter in the 2D case considered here. Thus, the mean-field shift is expected to be independent of the in-plane geometry of the cloud.

[1] U.R. Fischer, "Stability of quasi-two-dimensional Bose-Einstein condensates with dominant dipole-dipole interactions," Phys. Rev. A **73**, 031602 (2006).