Supplemental Material:
Coherent seeding of the dynamics of a spinor Bose-Einstein condensate: from quantum to classical behavior

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I. INITIAL STATE PREPARATION

A. Oscillating regime

We prepare the spinor BEC at $t = 0$ in a generalized coherent spin state $|\psi_{\rm ini}\rangle = (\sum_m \zeta_{{\rm ini},m}|m\rangle)^{\otimes N}$,

$$\boldsymbol{\zeta}_{\rm ini} = \begin{pmatrix} \sqrt{n_{\rm seed}}\mathrm{e}^{i\frac{\theta_{\rm ini}+\eta_{\rm ini}}{2}} \\ \sqrt{1-2n_{\rm seed}} \\ \sqrt{n_{\rm seed}}\mathrm{e}^{i\frac{\theta_{\rm ini}-\eta_{\rm ini}}{2}} \end{pmatrix}.$$

We prepare this state starting from $|m = 0\rangle$ using a combination of magnetic field ramps and resonant radio-frequency (rf) pulses. In details, we first pulse a rf field resonant with the Zeeman splitting to populate the $m = \pm 1$ modes with a fraction $n_{\rm seed} = \sin^2(\Omega_{\rm rf} t_1)/2$ of the atoms. Here, $\Omega_{\rm rf}$ is the rf Rabi frequency and t_1 the pulse duration. At this stage, we have prepared a coherent spin state of the form (1) with $\theta_{\rm ini} \approx \pi$.

To change $\theta_{\rm ini}$, we let the system evolve in a field $B = 0.5\,$G ($q/h \approx 70\,$Hz) for a time $t_2 < h/(2q)$, before quenching the magnetic field down to $28 \pm 2\,$mG ($q/h \approx 0.22\,$Hz) in $t_3 = 4\,$ms to achieve the desired regime $U_s/N \ll q \ll U_s$. Interactions are negligible ($U_s/h \approx 10\,$Hz hence $U_s t_{2,3}/h \ll 1$), and the system simply acquires a phase shift $\Delta\theta_2 = -2qt_2/\hbar$ while the magnetic field is held constant, and $\Delta\theta_3 = -2\int q(t)dt/\hbar$ during the quench. This results in an initial phase $\theta_{\rm ini} = \pi - 2qt_2/\hbar + \Delta\theta_3$ that is fully tunable from 0 to 2π by varying t_2.

B. Relaxing regime

We prepare mesoscopic BECs of $N \approx 124$ atoms in the same initial spin state as before. We lower the magnetic field down to $B = 4.2 \pm 1.5\,$mG ($q/h \approx 5\,$mHz) in $t_3 = 20\,$ms. The ramp time corresponds to the time needed for the damping of eddy currents in the vacuum chamber. Because of the small atom number, the effects of the spin dependent interactions are negligible over the ramp ($U_s/h \approx 4\,$Hz, such that $U_s t_3/h \ll 1$) and the evolution of the state is essentially another phase shift of θ, which can be compensated for by varying t_2. For these experiments, we always choose t_2 such that $\theta_{\rm ini} \approx 0$.

Finally, we trigger the dynamics by recompressing the trap in $6\,$ms ($U_s/h \approx 4 \to 24\,$Hz). By performing numerical simulations of the sequence with the many-body Schrödinger equation, we have checked that the ramp can be considered instantaneous to a good approximation.

II. CLASSICAL AND SEMI-CLASSICAL DYNAMICS

We detail here the calculations of the dynamics of $N_{\rm p}(t)$ given in the main text. We use a classical (C) approach based on the mean-field approximation and a semi-classical (SC) approach inspired by the truncated Wigner approximation (TWA). In both frameworks, the annihilation operators $\hat{a}_m$ are replaced by c-numbers $\alpha_m = \sqrt{N}\zeta_m$, with N the number of condensed atoms and $\boldsymbol{\zeta}$ a spin-1 wavefunction (normalized to unity) parameterized as

$$\boldsymbol{\zeta} = \begin{pmatrix} \sqrt{n_{\rm p}}\mathrm{e}^{i\frac{\theta+\eta}{2}} \\ \sqrt{1-2n_{\rm p}} \\ \sqrt{n_{\rm p}}\mathrm{e}^{i\frac{\theta-\eta}{2}} \end{pmatrix}. \tag{1}$$

Here $n_{\rm p} = (N_{+1} + N_{-1})/(2N)$ denotes the average number of $m = \pm 1$ pair normalized to the total atom number ($N_{\rm p} = Nn_{\rm p}$), and we have restricted ourselves to the situation $N_{+1} = N_{-1}$. We also have chosen ζ_0 real without loss of generality.

The mean field equations of motion for a spin-1 condensate in the single-mode regime take the form [1, 2]

$$\hbar\dot{n}_{\rm p} = -2U_s n_{\rm p}(1-2n_{\rm p})\sin\theta \tag{2}$$
$$\hbar\dot{\theta} = -2q + 2U_s(4n_{\rm p}-1)(1+\cos\theta)\,. \tag{3}$$

The mean-field energy per atom is given by

$$\mathcal{E}_s = 2U_s n_{\rm p}(1-2n_{\rm p})(1+\cos\theta) + 2qn_{\rm p}\,. \tag{4}$$

The energy $\mathcal{E}_s$ is a constant of motion, a fact that we will used repeatedly in the following.

A. Dynamics in the oscillating regime

In this section we derive the evolution of $N_{\rm p}(t)$ for the oscillating regime $q \gg U_s/N$. We assume $N_{\rm seed} \ll$

N, *i.e.* the situation where quantum fluctuations may play a significant role. For $N_{\rm seed} \sim N$, a fully classical treatment is accurate.

a. Classical solution : Assuming $n_{\rm p} \ll 1$, we linearize Eqs. (2) and (4),

$$\hbar \dot{n}_{\rm p} \approx -2U_s n_{\rm p} \sin\theta \tag{5}$$

$$\mathcal{E}_s \approx \Big(2U_s(1+\cos\theta) + 2q\Big) n_{\rm p}\,. \tag{6}$$

We use the second equation to express $\cos\theta$ as a function of $n_{\rm p}$ and of the constants $q, U_s, \mathcal{E}_s$. Substituting in the first equation, we obtain a differential equation on $n_{\rm p}$ only, $\dot{n}_{\rm p}^2 = -4\omega^2\left[n_{\rm p}-\alpha\right]^2 + A$, where

$$\hbar\omega = \sqrt{q(q+2U_s)}, \qquad \alpha = \frac{\mathcal{E}_s(q+U_s)}{2(\hbar\omega)^2}, \tag{7}$$

and where A is constant. Differentiating one more time, we find that either $n_{\rm p}$ is constant or it obeys the harmonic equation $\ddot{n}_{\rm p} + 4\omega^2\left(n_{\rm p}-\alpha\right) = 0$. The evolution is thus a harmonic motion at frequency 2ω,

$$n_{\rm p}(t) \approx n_{\rm seed} + 2(\alpha - n_{\rm seed})\sin^2(\omega t)\,, \tag{8}$$

with the initial conditions $n_{\rm p}(0) = n_{\rm seed}$ and $\theta(0) = \theta_{\rm ini}$.

If we further assume (as in the experiments we performed) that $q \ll U_s$, we have $\mathcal{E}_s \approx 4U_s n_{\rm seed}\cos^2(\theta_{\rm ini}/2) \gg q$, and $\alpha \approx \mathcal{E}_s/(4q) \gg 1$. Eq. (8) then reduces to

$$n_{\rm p}(t) \approx n_{\rm seed} + \frac{2U_s n_{\rm seed}}{q}\,\cos^2(\theta_{\rm ini}/2)\,\sin^2(\omega t)\,,$$

i.e. to Eq. (6) in the main text.

b. Semi-classical picture : We now consider the effect of quantum fluctuations within the TWA [3–7]. In this method, the c-numbers α_m used instead of the annihilation operators $\hat{a}_m$ in the mean-field approximation are treated as complex random variables. At $t=0$, these variables sample the Wigner distribution of the initial state $|\psi_{\rm i}\rangle$. Their mean values are given by

$$\bar{\boldsymbol{\alpha}}_{\rm ini} = N \begin{pmatrix} \sqrt{n_{\rm seed}}\,{\rm e}^{i\frac{\theta_{\rm ini}+\eta_{\rm ini}}{2}} \\ \sqrt{1-2n_{\rm seed}} \\ \sqrt{n_{\rm seed}}\,{\rm e}^{i\frac{\theta_{\rm ini}-\eta_{\rm ini}}{2}} \end{pmatrix}. \tag{9}$$

In the limit $N_{\rm seed} \ll N$, the calculation can be simplified by neglecting the depletion of the mode $m=0$. For the $m=\pm1$ modes, this approximation amounts to replacing coherent spin states by harmonic oscillator coherent states, which are considerably easier to handle. The initial quantum state is thus taken to be

$$|\psi_{\rm ini}\rangle \approx \frac{1}{\sqrt{N!}}\prod_{m=\pm1} {\rm e}^{\bar{\alpha}_{m,{\rm ini}}\hat{a}_m^\dagger - \bar{\alpha}^*_{m,{\rm ini}}\hat{a}_m}\,\hat{a}_0^{\dagger N}|{\rm vac}\rangle\,. \tag{10}$$

For $t>0$, the equations of evolution (2,3) remain valid in the TWA. The solution for initial conditions $\alpha_{\pm1,{\rm ini}}$ is thus given by Eq. (8) with the substitution $4N_{\rm seed}\cos^2(\theta_{\rm ini}/2) \to |\alpha_{+1,{\rm ini}} + \alpha^*_{-1,{\rm ini}}|^2$.

To average over the initial distribution of $\alpha_{\pm1,{\rm ini}}$, we recall that the Wigner distribution average $\langle \mathcal{O}(\alpha_m,\alpha_m^*)\rangle_{\rm Wig}$ of an operator $\mathcal{O}$ is equal to the expectation value $\langle \mathcal{O}^{\rm sym}(\hat{a}_m,\hat{a}_m^\dagger)\rangle$ of the corresponding symmetrically ordered operator $\mathcal{O}^{\rm sym}$ [3]. We obtain

$$\langle \alpha_{+1,{\rm ini}}\alpha^*_{-1,{\rm ini}}\rangle_{\rm Wig} = \langle \hat{a}_{+1}\hat{a}^\dagger_{-1}\rangle = \bar{\alpha}_{+1,{\rm ini}}\bar{\alpha}^*_{-1,{\rm ini}}\,, \tag{11}$$

$$\langle |\alpha_{m,{\rm ini}}|^2\rangle_{\rm Wig} = \frac{1}{2}\langle \hat{a}_m^\dagger\hat{a}_m + \hat{a}_m\hat{a}_m^\dagger\rangle = |\bar{\alpha}_{m,{\rm ini}}|^2 + \frac{1}{2}\,. \tag{12}$$

This leads to

$$\langle N_{\rm p}(t)\rangle \approx \frac{U_s}{2q}\sin^2(\omega t)\,\left(|\bar{\alpha}_{+1,{\rm ini}} + |\bar{\alpha}^*_{-1,{\rm ini}}|^2 + 1\right)\,,$$

which gives Eq. (7) in the main text.

As a final remark, we note that the Bogoliubov method is also well suited to study the regime that we investigated here, and leads to the same result [8–10].

B. Relaxation dynamics

We now discuss the regime $q \ll U_s/N$, in which we observe a relaxation of the number of pairs $N_{\rm p}$ to a stationary value. In this regime, the quantum fluctuations play an important role even for $N_{\rm seed} \gg 1$. We will thus consider that $N_{\rm seed} \gg 1$ and $N - N_{\rm seed} \gg 1$. For simplicity, we will focus on the situation $\theta_{\rm ini} = 0$, for which the effect of the seed is maximal. The case with no seed has been treated using an exact diagonalization of the Hamiltonian [10] or the TWA [6].

a. Classical solution In order to simplify the calculation, we neglect completely the quadratic Zeeman shift. In this regime $q \ll U_s/N$, the Zeeman term indeed plays no significant role even for the fully quantum model. Introducing the auxiliary variable $x = 4n_{\rm p} - 1$, the equations of motion and the energy become

$$\hbar\dot{x} = -U_s(1-x^2)\sin\theta\,, \tag{13}$$

$$\hbar\dot{\theta} = 2U_s x(1+\cos\theta)\,, \tag{14}$$

$$\mathcal{E}_s = \frac{U_s}{4}(1-x^2)(1+\cos\theta) = {\rm cst}\,. \tag{15}$$

We combine the first and last equations to obtain

$$\dot{x} = -\frac{4\mathcal{E}_s}{\hbar}\frac{\sin\theta}{1+\cos\theta}. \tag{16}$$

Differentiating this equation, we eliminate the phase θ and obtain a simple harmonic equation, $\ddot{x} = -\Omega^2 x$, with an oscillation frequency $\hbar\Omega = \sqrt{8U_s\mathcal{E}_s}$. For the initial conditions $n_{\rm p}(0) = n_{\rm seed}$ and $\theta(0) = 0$, we have $\hbar\Omega = 2U_s\sqrt{1-x_0^2}$ and $x(t) = x_0\cos(\Omega t)$ with $x_0 = 4n_{\rm seed} - 1$. This corresponds to the results announce in Eqs. (8,9) of the main text.

b. Quantum partition noise: The initial state

$$|\psi_{\rm ini}\rangle = \frac{1}{\sqrt{N!}}\left[\sum_{m=0,\pm1}\zeta_m\,\hat{a}_m^\dagger\right]^N|{\rm vac}\rangle\,,$$

is characterized by fluctuations of the number of ±1 atoms. We consider again the states with $|\zeta_{+1}| = |\zeta_{-1}| = \sqrt{N_{\rm seed}}$ and $\theta_{\rm i} = 0$. We introduce the sum $\Sigma = N_{+1} + N_{-1}$, its relative value $s = \Sigma/N$ and the difference $\Delta = N_{+1} - N_{-1}$. The components of $\boldsymbol{\zeta}$ are related to the average $\bar{\Sigma}$ of Σ by

$$|\zeta_{\pm1}|^2 = \frac{\bar{\Sigma}}{2}, \quad |\zeta_0|^2 = N - \bar{\Sigma}. \tag{17}$$

The joint distribution of Σ and Δ in the initial coherent spin state is

$$\mathcal{P}(\Sigma,\Delta) = \frac{N!}{\left(\frac{\Sigma+\Delta}{2}\right)!\left(\frac{\Sigma-\Delta}{2}\right)!(N-\Sigma)!}\left(\frac{\bar{s}}{2}\right)^{\Sigma}(1-\bar{s})^{N-\Sigma}. \tag{18}$$

We deduce from Eq. (18) the marginal distribution of Σ,

$$\mathcal{P}(\Sigma) = \frac{N!}{\Sigma!(N-\Sigma)!}\bar{s}^{\Sigma}(1-\bar{s})^{N-\Sigma}. \tag{19}$$

with $\Sigma \in [0, N]$. The normalization follows from the binomial formula.

For large N and Σ away from the extreme values $0, N$, the binomial distribution is well approximated by a continuous Gaussian distribution

$$\mathcal{P}(\Sigma) \approx \frac{1}{N}\frac{1}{\sqrt{2\pi}\sigma}{\rm e}^{-\frac{(s-\bar{s})^2}{2\sigma^2}} = p(s)ds. \tag{20}$$

with a step size $ds = 1/N$ and a standard deviation

$$\sigma = \sqrt{\frac{\bar{s}(1-\bar{s})}{N}} = \sqrt{\frac{2n_{\rm seed}(1-2n_{\rm seed})}{N}}. \tag{21}$$

One can check the normalization of both distributions,

$$\sum_{\Sigma=0}^{N}\mathcal{P}(\Sigma) \to \int_0^1 f(s)ds \approx \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}{\rm e}^{-\frac{u^2}{2}}du = 1.$$

To extend the lower boundary to $-\infty$, we require $\bar{s}/\sigma = \sqrt{N}\times\sqrt{\bar{s}/(1-\bar{s})} \gg 1$, or $N\bar{s} = 2N_{\rm seed} \gg 1$.

c. Semi-classical picture of the dynamics: Similarly to what we have done in Sec. II A, we average the mean field solution (2,3) with $2n_{\rm seed} \to s$ over the probability distribution $p(s)$ in Eq. (20). This amounts to compute the integral

$$I = \frac{1}{2}\int_0^1 s\,\cos[\Omega(s)t]\,p(s)\,ds. \tag{22}$$

We use the fact that $p(s)$ is sharply peaked around $\bar{s}$, with a width $\sim 1/N$ much narrower than the scale of variation of the rest of the integrand $s\cos[\Omega(s)t]$. As a result, we extend the integral boundaries to $\pm\infty$, set $s \approx \bar{s}$ and expand the frequency $\Omega(s)$ to first order,

$$\Omega(s) \approx \bar{\Omega} + \bar{\Omega}'(s-\bar{s}) + \mathcal{O}(\epsilon^2)\,, \tag{23}$$

where $\bar{\Omega} = \Omega(\bar{s})$ and $\bar{\Omega}' = \Omega'(\bar{s}) = (2U_s/\hbar)\times(1-2\bar{s})/\sqrt{\bar{s}(1-\bar{s})}$.

With straightforward manipulations, we cast I in the form of the Fourier transform of a Gaussian function, which is readily calculated. We find

$$I = \frac{1}{2}\bar{s}\cos[\bar{\Omega}t]\,{\rm e}^{-\frac{1}{2}(\gamma_{\rm c}t)^2}, \tag{24}$$

with a damping rate

$$\gamma_{\rm c} = |\bar{\Omega}'\sigma| = \frac{2U_s}{\sqrt{N}\hbar}|1-2\bar{s}|. \tag{25}$$

Using $\bar{s} = 2n_{\rm seed}$, this gives Eq. (11) in the main text.

d. Classical fluctuations of Ω*:* In addition to the intrinsic dephasing originating from quantum fluctuations, any technical fluctuations of Ω will also contribute to the observed relaxation. We consider here the dominant source of classical blurring in our experiment, namely fluctuations of the interaction strength U_s mainly due to shot-to-shot atom number fluctuations.

We model these fluctuations by considering a fluctuating interaction strength $U_s' = U_s + \delta U_s x$, with U_s the average value, δU_s the standard deviation of the noise, and x a centered Gaussian random variable of variance unity. This leads to a fluctuating oscillation frequency $\Omega(x) = \bar{\Omega}(1 + x\cdot\delta U_s/U_s)$. We neglect the fluctuations of $\gamma_{\rm c}$, which is legitimate for $N_{\rm seed} \gg 1$ and hence $\gamma_{\rm c} \ll \bar{\Omega}$. Averaging over the Gaussian probability distribution $p(x)$, we find that

$$I_2 = \left\langle\cos[\Omega(x)t]{\rm e}^{-\frac{1}{2}(\gamma_{\rm c}t)^2}\right\rangle_x = \cos[\bar{\Omega}t]e^{-\frac{1}{2}(\gamma_{\rm t}t)^2-\frac{1}{2}(\gamma_{\rm c}t)^2}, \tag{26}$$

with a classical (technical) damping rate given by

$$\gamma_{\rm t} = \bar{\Omega}\frac{\delta U_s}{U_s}. \tag{27}$$

From Eqs. (26,27) we obtain Eqs. (12,13) given in the main text.

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