Supplementary Material for Dynamic optical lattices of sub-wavelength spacing for ultracold atoms

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S.I. EFFECTIVE POTENTIAL CREATED BY THE STROBOSCOPIC SCHEME

In this Section, we present the calculation of the effective lattice potential produced via the stroboscopic scheme illustrated in Fig. 1 (main text). We start from a periodic potential $V(x)$ of period d, which can be decomposed in Fourier series as

$$V(x) = \sum_{p\in\mathbb{Z}} V_p\, e^{i2\pi px/d}.$$

The method consists in shifting the potential $V(x)$ of the distance d/N, after each time interval T/N. For N integer, this leads to a time-periodic potential $V(x,t)$ of time period T. For a sufficiently short period T, the atomic motion is governed by the effective potential $V_{\rm eff}(x)$, equal to the time average of $V(x,t)$:

$$\begin{aligned} V_{\rm eff}(x) &= \frac{1}{T}\int_0^T V(x,t)\mathrm{d}t \\ &= \frac{1}{N}\sum_{j=1}^{N} V(x+jd/N) \\ &= \sum_{p\in\mathbb{Z}} V_p\, e^{i2\pi px/d}\frac{1}{N}\sum_{j=1}^{N} e^{i2\pi pj/N} \\ &= \sum_{p \text{ multiple of } N} V_p\, e^{i2\pi px/d}. \end{aligned}$$

It is then apparent that the effective potential $V_{\rm eff}(x)$ is periodic, of period $d_{\rm eff} = d/N$.

S.II. CALCULATION OF THE EFFECTIVE POTENTIAL

In this section we give details about the establishment of the effective Hamiltonian (eqs. (3), (4) in the main text). We first provide a simple description of the effective Hamiltonian in a Born-Oppenheimer approximation, and discuss further the relevance of this approximation for typical lattice parameters.

The Born-Oppenheimer approximation consists in neglecting the kinetic energy for the calculation of the effective Hamiltonian. The position x is considered as a fixed parameter, while internal degrees of freedom are treated quantum mechanically. Using the formalism of ref. [S1], we decompose the Hamiltonian in Fourier series as

$$\begin{aligned} H(t) &= H_0 + \sum_{j=1}^{\infty} V^{(j)}e^{ij\Omega t} + V^{(-j)}e^{-ij\Omega t}, \\ V^{(1)} &= V^{(-1)\dagger} = V_{\rm L}e^{-2ikx}\sigma_z/2, \\ V^{(N)} &= V^{(-N)\dagger} = V_{\rm B}\sigma_x/2, \\ V^{(j)} &= 0 \quad \text{otherwise.} \end{aligned}$$

The effective Hamiltonian can be expanded as a series in $1/\Omega$. At lowest order, it reads

$$\begin{aligned} H_{\rm eff} &= \frac{p^2}{2m} + V_{\rm eff}(x), \\ V_{\rm eff}(x) &= \frac{1}{N!(\hbar\Omega)^N}\left[V^{(1)},\ldots,\left[V^{(1)},V^{(-N)}\right]\right] + \text{h.c.}, \end{aligned}$$

with $V^{(1)}$ occurring N times. One calculates

$$\begin{aligned} V_{\rm eff}(x) &= \frac{V_{\rm L}^N V_{\rm B} e^{-i2Nkx}}{2N!(2\hbar\Omega)^N}\left[\sigma_z,\ldots,[\sigma_z,\sigma_x]\right] + \text{h.c.} \\ &= \frac{U_{\rm eff}}{2}\begin{cases} \cos(2Nkx)\sigma_x, & N \text{ even}, \\ \sin(2Nkx)\sigma_y, & N \text{ odd}, \end{cases} \end{aligned}$$

where $U_{\rm eff} = 2V_{\rm B}(V_{\rm L}/\hbar\Omega)^N/N!$.

In the main text we consider the case of a (pseudo)-spin-1/2 for simplicity. The scheme can directly be extended to an arbitrary spin F. Assuming a coupling $V(x,t) = 2V_{\rm L}\cos(2kx-\Omega t)F_z + 2V_{\rm B}\cos(N\Omega t)F_x$, we make use of the general commutation algebra of F_u operators and obtain an effective potential $V_{\rm eff}(x) = U_{\rm eff}\cos(2Nkx)F_x$ (N even).

We now consider the validity of the Born-Oppenheimer approximation, which consists in neglecting the non commutation of position and momentum. The latter plays a role at the order $N+1$ of the perturbative expansion

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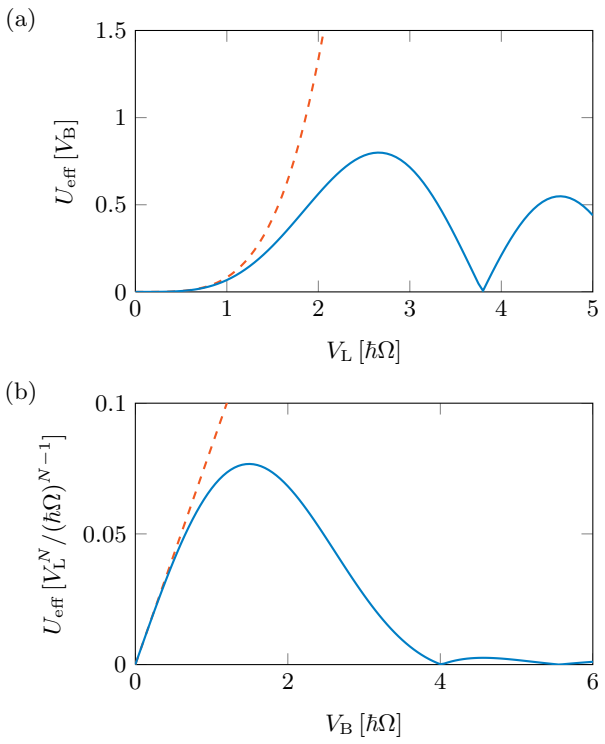

FIG. S1. Depth $U_{\rm eff}$ of the effective lattice, calculated for arbitrary values of $V_{\rm L}/(\hbar\Omega)$ (a) or $V_{\rm B}/(\hbar\Omega)$ (b) in the case $N = 4$. The dashed lines correspond to the lowest-order perturbation result (4), and the solid lines to the resummation results (S.1) and (S.2).

in $1/\Omega$. We performed the perturbative expansion up to the order $N + 1$ for the case $N = 1$, and obtained the expression

$$H_{\rm eff} = \frac{p^2}{2m} + \frac{U_{\rm eff}}{2}\sin(2kx)\sigma_y + \left(\frac{V_{\rm L}}{\hbar\Omega}\right)^2 E_{\rm r} + \frac{U_{\rm eff}}{2}\frac{\hbar k}{2m\Omega}\left(p\sin(2kx) + \sin(2kx)p\right)\sigma_y.$$

While the first term of order $N+1$ only represents an energy offset, the second term can be viewed as a spin-orbit coupling which may affect the atom dynamics. However, for typical momenta p on the order of the lattice momentum k, the amplitude of this term is smaller than the effective lattice depth $U_{\rm eff}$ by a factor $\sim E_{\rm r}/(\hbar\Omega)$, which is typically very small for the examples considered in this Letter.

S.III. RESUMMATION OF THE PERTURBATIVE EXPANSION OF $H_{\rm eff}$

The effective potential $V_{\rm eff}$ can be calculated as a series expansion in powers of the (potentially small) dimensionless parameters $V_{\rm L}/(\hbar\Omega)$ and $V_{\rm B}/(\hbar\Omega)$, in the high-frequency limit $\Omega \to \infty$. In the main text, we provide its expression in Eq. (4), which corresponds to the lowest-order term. We note that this derivation, which is based on the general formula of Ref. [S1], was obtained by neglecting the non-commutativity of the position and momentum operators; indeed, we verified that the momentum operator is irrelevant in the derivation of the effective potential, which essentially relies on the spin-dependent time-modulated components of the Hamiltonian. Thus, in the following of this Section, which aims to derive the effective potential in the strong-modulation regime, we explicitly neglect any effects associated with the kinetic energy term of the full Hamiltonian.

In this Section, we first derive the expression for the effective potential $V_{\rm eff}$, in the case where $V_{\rm L}/(\hbar\Omega)$ is allowed to take arbitrary large values (still assuming that $V_{\rm B} \ll V_{\rm L}, \hbar\Omega$). Following Refs. [S2, S3], we perform a unitary transformation

$$|\psi'\rangle = R(t)\,|\psi\rangle\,, \quad R(t) = \exp\left(-i\frac{V_{\rm L}}{\hbar\Omega}\sin(kx - \Omega t)\sigma_z\right),$$

which removes the diverging term $\sim V_L \sim \hbar\Omega$ from the time-dependent potential $V(x,t)$ in Eq. 1 (main text). This leads to a novel time-dependent potential

$$\begin{aligned} V'(x,t) &= R(t)V(x,t)(t)R^\dagger(t) + i\hbar\partial_t R(t)R^\dagger(t) \\ &= R(t)\left[V_{\rm B}\cos(N\Omega t)\sigma_x\right]R^\dagger(t). \end{aligned}$$

Making use of the identity $e^{-i\gamma\sigma_z}\sigma_x e^{i\gamma\sigma_z} = \cos(2\gamma)\sigma_x + \sin(2\gamma)\sigma_y$, we obtain the expression

$$V'(x,t) = V_{\rm B}\cos(N\Omega t)\left[\cos\left(\frac{2V_{\rm L}}{\hbar\Omega}\sin(kx - \Omega t)\right)\sigma_x + \sin\left(\frac{2V_{\rm L}}{\hbar\Omega}\sin(kx - \Omega t)\right)\sigma_y\right].$$

In the large-frequency limit $\Omega \to \infty$, the atom dynamics can be described by an effective stationary potential, given by [S2, S3]

$$\begin{aligned} V_{\rm eff}(x) &= \frac{1}{T}\int_0^T V'(x,t)\mathrm{d}t \\ &= J_N\left(\frac{2V_{\rm L}}{\hbar\Omega}\right)V_{\rm B}\,\cos(2Nkx)\sigma_x, \end{aligned} \tag{S.1}$$

assuming N even, and where J_N is a Bessel function of the first kind. This effective potential corresponds to a spin-dependent optical lattice of spacing d/N and depth $U_{\rm eff} = 2J_N\left(\frac{2V_{\rm L}}{\hbar\Omega}\right)V_{\rm B}$.

A similar resummation with respect to $V_{\rm B}/(\hbar\Omega)$ can also be derived. Here, we make use of the Floquet representation of time-periodic Hamiltonians. We first write the exact eigenstates of the coupling $\frac{V_{\rm B}}{2}\cos(N\Omega t)\sigma_x$, which read

$$||n, s_x\rangle = \sum_{p\in\mathbb{Z}} J_p\left(\frac{2s_x V_{\rm B}}{\hbar\Omega}\right)|n + pN, s_x\rangle,$$

where n denotes the Floquet quantum number, and s_x is the spin projection along x. The energy of the state $||n, s_x\rangle$ is equal to $n\hbar\Omega$. The effect of the coupling $V_{\rm L}\cos(kx-\Omega t)\sigma_x$ can be understood using perturbation theory in the degenerate subspace $||n,\pm\rangle$, which must be performed at order N. We obtain the expression

$$V_{\rm eff}(x) = \frac{U_{\rm eff}}{2}\cos(2Nkx)\sigma_x,$$
$$U_{\rm eff} = 4\hbar\Omega\left(\frac{V_{\rm L}}{2\hbar\Omega}\right)^N \left|\sum_{\sum_{i=1}^N p_i=-1} \frac{\prod_{i=1}^N J_{p_i}[(-1)^i\frac{2V_{\rm B}}{N\hbar\Omega}]}{\prod_{i=1}^{N-1}\sum_{j=1}^i(1+Np_j)}\right|. \quad \text{(S.2)}$$

We plot in Fig. S1 the lattice depth $U_{\rm eff}$ given by the resummation formulas in Eqs. (S.1)-(S.2) discussed above. We checked that the formulas (S.1) and (S.2) account well for the numerical results obtained via direct diagonalization of the Bloch-Floquet equations (see Section S.IV).

S.IV. EXPRESSION FOR THE BLOCH-FLOQUET HAMILTONIAN

The modulated potential (1) is invariant under the space and time translational symmetries $\mathcal{T}_x$, $\mathcal{T}_t$ and $\mathcal{T}^*$, which all commute with each other. The eigenstates of the Hamiltonian can thus be written as eigenstates of those symmetries, which can be expressed as

$$\psi_{\tilde{q},\omega}(x,t) = e^{i(\tilde{q}x-\omega t)}\sum_{j,l\in\mathbb{Z}} c_{j,l}\, e^{il(kx-\Omega t)}e^{ijNkx},$$

where $-Nk < \tilde{q} \leq Nk$ and $0 \leq \omega < \Omega$. The spinor coefficients $c_{j,l}$ are determined by the equations

$$\begin{aligned}\hbar(\omega+l\Omega)c_{j,l} =& \frac{\hbar^2[\tilde{q}+(l+Nj)k]^2}{2m}c_{j,l} \\ &+ \frac{V_{\rm L}}{2}\sigma_x(c_{j,l+1}+c_{j,l-1}) \\ &+ \frac{V_{\rm B}}{2}\sigma_z(c_{j+1,l-N}+c_{j-1,l+N}).\end{aligned}$$

The numerical data represented in Fig. 2 (main text) is calculated using the above equations, in a truncated basis $-10 \leq j,l \leq 10$.

S.V. MICRO-MOTION EFFECTS IN THE MOMENTUM DISTRIBUTION

In this section, we analyze how the micro-motion associated with the time-modulation in Eq. (1) affects the momentum distribution of atoms prepared in the effective potential $V_{\rm eff}$ of spatial period d/N [Eq. (4)]. Specifically, we consider an atom prepared in the ground state of the effective potential. This state can be expanded on the family of states of momentum multiple of $2Nk$ (see Fig. S2a).

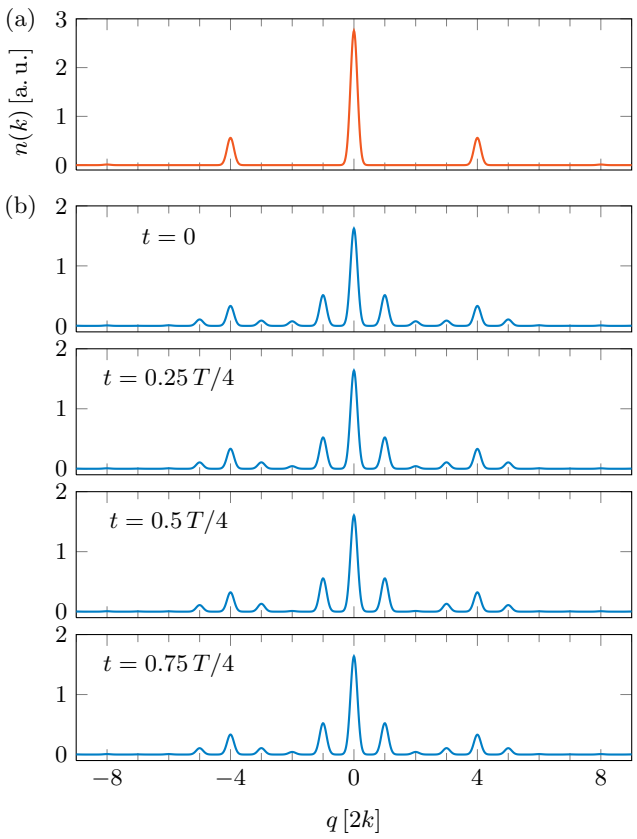

FIG. S2. (a) Momentum distribution associated with the ground state of the effective lattice with spacing $d/4$ and depth $U_{\rm eff} = 10.9\,E_{\rm r}^{\rm eff}$. (b) Momentum distribution of the state in (a), taking into account the micro-motion expected for the dynamic lattice parameters, according to Eq. S.3. The micro-motion leads to a more complex structure compared to (a), periodically evolving in time. The lattice parameters correspond to the ones of Fig. 2 in the main text.

The actual state created using time-modulated lattices is expected to be modified by the micro-motion, as

$$|\psi(t)\rangle = e^{-iK(t)}\,|\psi_0\rangle\,, \quad \text{(S.3)}$$

where the expression for the kick operator $K(t)$ is given in the main text [Eq. (5)]. The latter leads to additional diffraction peaks at all momenta multiple of $2k$, whose amplitude vary periodically in time, with a period T/N (see Fig. S2b). This shows that Bragg diffraction does not give a direct information on the ground state of the effective lattice.

S.VI. EFFECTIVE HAMILTONIAN DURING LATTICE LOADING

In this Section, we analyze the adiabatic preparation of the ground state associated with the effective potential

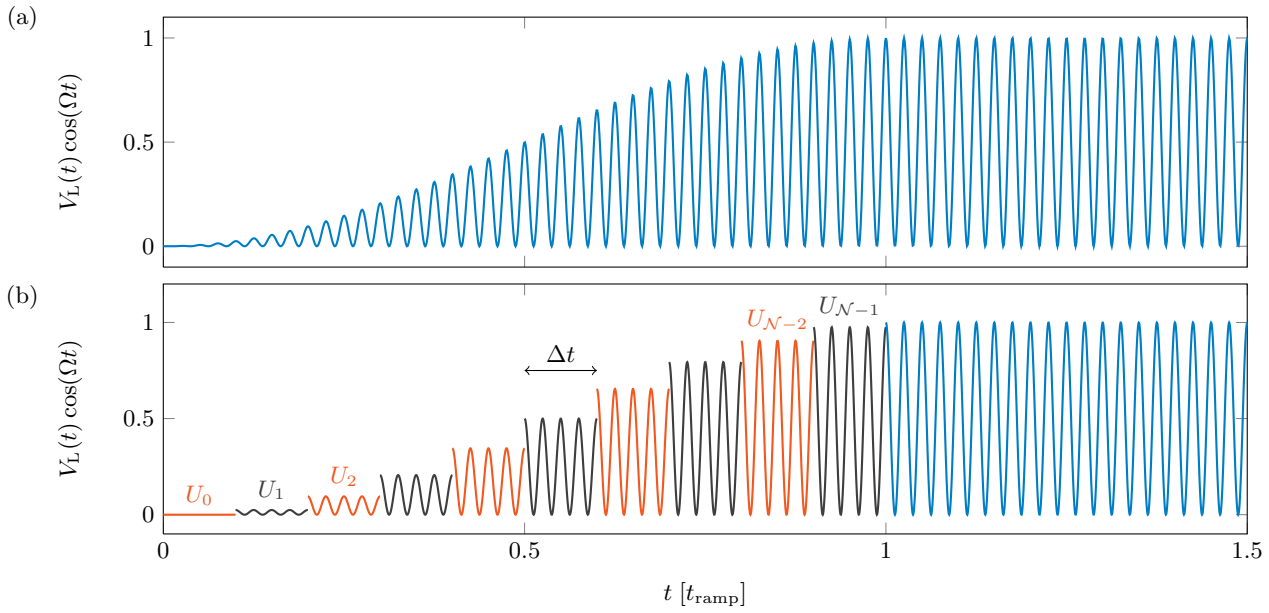

FIG. S3. (a) Evolution of the amplitude of the moving optical lattice at position $x = 0$, during and after the lattice ramp of duration $t_{\rm ramp}$. (b) Scheme of the ramp discretization: the time interval $0 \leq t \leq t_{\rm ramp}$ is decomposed into $\mathcal{N}$ steps of duration Δt. Within each step the depth $V_{\rm L}$ is constant, leading to a time-periodic potential.

$V_{\rm eff}(x)$ of depth $U^0_{\rm eff}$. We consider a slow ramp of the moving-lattice depth $V_{\rm L}(t) = 0 \to V^0_{\rm L}$ during the time interval $0 \leq t \leq t_{\rm ramp}$, such that the effective potential's depth $U^0_{\rm eff}$ corresponds to the final value $V_{\rm L}(t_{\rm ramp}) = V^0_{\rm L}$. As the definition (2) of the effective Hamiltonian and kick operators assumes a constant lattice depth [S1], we expect these notions to be modified during the ramp. It is the aim of this Section to show how the adiabatic ramp can still be captured by an effective-Hamiltonian picture.

To analyze this situation, we decompose the ramp into $\mathcal{N}$ steps, and we assume that the time interval $\Delta t = t_{\rm ramp}/\mathcal{N}$ is short enough, such that $V_{\rm L}$ can be considered to remain constant within each step. More precisely, we assume that the lattice depth is equal to $V_{\rm L}(j\Delta t)$ during the step $j\Delta t \leq t < (j+1)\Delta t$. We then apply the effective-Hamiltonian formalism of Ref. [S1] within each time-step, and write the full time-evolution operator as

$$U_{\rm ramp} = \prod_{j=\mathcal{N}-1}^{0} U_j,$$
$$U_j = e^{-iK_0[V_{\rm L}(j\Delta t)]} e^{-iH_{\rm eff}[V_{\rm L}(j\Delta t)]\Delta t/\hbar} e^{iK_0[V_{\rm L}(j\Delta t)]}.$$

In the latter expression, and for the sake of simplicity, we assumed that Δt was a multiple of the modulation period, so that the kick operators at the beginning and at the end of each step only depend on the value of $V_{\rm L}$ (in fact, they correspond to the kick operator at the time $t = 0$, hence the notation K_0).

Assuming Δt short enough, we write

$$e^{iK_0[V_{\rm L}((j+1)\Delta t)]} e^{-iK_0[V_{\rm L}(j\Delta t)]} \simeq e^{i\Delta t(\mathrm{d}V_{\rm L}/\mathrm{d}t)\mathrm{d}K_0/\mathrm{d}V_{\rm L}},$$

leading to

$$U_{\rm ramp} = e^{-iK(t_{\rm ramp})} \mathcal{T}\left\{\exp\left(-i\int H^{\rm ramp}_{\rm eff}(t)\mathrm{d}t/\hbar\right)\right\},$$

where $\mathcal{T}$ denotes time-ordering, and where one introduced the slowly varying Hamiltonian

$$\begin{aligned} H^{\rm ramp}_{\rm eff}(t) &= H_{\rm eff}|_{V_{\rm L}(t)} - \hbar \frac{\mathrm{d}V_{\rm L}}{\mathrm{d}t} \left.\frac{\mathrm{d}K_0(t_{\rm ramp})}{\mathrm{d}V_{\rm L}}\right|_{V_{\rm L}(t)} \\ &= \frac{U_{\rm eff}(t)}{2}\cos(2Nkx)\sigma_x + \frac{1}{\Omega}\frac{\mathrm{d}V_{\rm L}}{\mathrm{d}t}\sin(2kx)\sigma_z. \end{aligned} \tag{S.4}$$

We now estimate a criterion for identifying the adiabatic regime of the lattice loading. Describing the lattice loading solely with the first term of eq. (S.4) would lead to the standard adiabaticity criterion $t_{\rm ramp} \gg \hbar U^0_{\rm eff}/(E^{\rm eff}_{\rm r})^2$ [S4]. We expect the second term in (S.4) to drive non-adiabatic transitions for $\dot{V}_{\rm L} \gtrsim \Omega E_{\rm r}$. Adiabatic lattice loading thus requires the additional constraint $t_{\rm ramp} \gg (V^0_{\rm L}/E_{\rm r})/\Omega$. As we choose $V_{\rm L} \lesssim \hbar\Omega$, this constraint should not be the most restrictive.

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